Unkonventionelle Supraleitung WS 05/06
Lösungen zur Serie S1

Multiple superconducting phase transitions.

S1 First of all, we consider a system with tetragonal crystal structure and strong spin-orbit coupling. For the two-component order parameter $\vec{\eta} = (\eta_x, \eta_y)$ corresponding to two basis functions of the two dimensional representation, the Ginzburg-Landau free energy density f is given as follows.

$$\begin{aligned} f &= a'(T - T_\mathrm{c})|\vec{\eta}|^2 + b_1|\vec{\eta}|^4 + \frac{b_2}{2}\{\eta_x^{*2}\eta_y^2 + \eta_x^2\eta_y^{*2}\} + b_3|\eta_x|^2|\eta_y|^2 && (1) \\ &= a'(T - T_\mathrm{c})\{|\eta_x|^2 + |\eta_y|^2\} + b_1\Big|\{|\eta_x|^2 + |\eta_y|^2\}\Big|^2 + \frac{b_2}{2}\{\eta_x^{*2}\eta_y^2 + \eta_x^2\eta_y^{*2}\} + b_3|\eta_x|^2|\eta_y|^2 && (2) \\ &= a'(T - T_\mathrm{c})\{|\eta_x|^2 + |\eta_y|^2\} + b_1\Big\{|\eta_x|^4 + |\eta_y|^4 + 2|\eta_x|^2|\eta_y|^2\Big\} + \frac{b_2}{2}\{\eta_x^{*2}\eta_y^2 + \eta_x^2\eta_y^{*2}\} + b_3|\eta_x|^2|\eta_y|^2. && (3) \end{aligned}$$

Here, we have omitted the gradient terms and the magnetic field term assuming a spatially uniform system under zero magnetic field. The coefficients a' (> 0) and b_i $(i = 1, 2, 3)$ are real numbers.

Now, let us consider the situation that the symmetry of the above system is reduced from the tetragonal symmetry (D_{4h}) to the orthorhombic one (D_{2h}). (That is, the x and y directions become inequivalent.) The symmetry reduction lifts the degeneracy of the two components η_x and η_y, leading to different transition temperatures for η_x and η_y.

The second order term of f becomes

$$a'(T - T_\mathrm{c})\{|\eta_x|^2 + |\eta_y|^2\} \quad \rightarrow \quad a'(T - T_{\mathrm{c}x})|\eta_x|^2 + a'(T - T_{\mathrm{c}y})|\eta_y|^2. \tag{4}$$

Here, we assume that the degeneracy lifting is small such that $T_{\mathrm{c}x}$ and $T_{\mathrm{c}y}$ are only slightly different, i.e., $|T_{\mathrm{c}x} - T_{\mathrm{c}y}| \ll T_{\mathrm{c}x}, T_{\mathrm{c}y}$.

Assuming $T_{\mathrm{c}x} > T_{\mathrm{c}y}$, the first superconducting transition occurs at $T = T_{\mathrm{c}x}$. That is, $\vec{\eta} = (0, 0) \rightarrow \vec{\eta} = (\eta_x, 0)$ at $T = T_{\mathrm{c}x}$. Then, the second transition from this phase $\vec{\eta} = (\eta_x, 0)$ to lower-temperature phase $\vec{\eta} = (\eta_x, \eta_y)$, occurs at $T = T'_{\mathrm{c}y}$ $(< T_{\mathrm{c}x})$.

In summary:

- $\vec{\eta} = (0, 0)$ at $T > T_{\mathrm{c}x}$
- $\vec{\eta} = (\eta_x, 0)$ at $T_{\mathrm{c}x} \geq T > T'_{\mathrm{c}y}$
- $\vec{\eta} = (\eta_x, \eta_y)$ at $T \leq T'_{\mathrm{c}y}$

For $T > T'_{cy}$ ($\eta_y = 0$), Eqs. (3) and (4) lead to

$$f = a'(T - T_{cx})|\eta_x|^2 + b_1|\eta_x|^4. \tag{5}$$

As usual (considering $\partial f/\partial \eta_x^* = 0$), this has a minimum at

$$|\eta_x(T)|^2 = \frac{-a'(T - T_{cx})}{2b_1}, \tag{6}$$

when $b_1 > 0$ and $T < T_{cx}$.

For $T < T'_{cy}$ ($\eta_y \neq 0$), Eqs. (3) and (4) lead to

$$\begin{aligned} f &= a'(T - T_{cx})|\eta_x|^2 + a'(T - T_{cy})|\eta_y|^2 \\ &\quad + b_1\Big\{|\eta_x|^4 + |\eta_y|^4 + 2|\eta_x|^2|\eta_y|^2\Big\} + \frac{b_2}{2}\{\eta_x^{*2}\eta_y^2 + \eta_x^2\eta_y^{*2}\} + b_3|\eta_x|^2|\eta_y|^2. \end{aligned} \tag{7}$$

It is known that T'_{cy} *is different from* T_{cy}. In what follows, let us calculate this second transition temperature T'_{cy}.

Let us recall the Ginzburg-Landau free energy with a single component order parameter ψ:

$$f = A|\psi|^2 + B|\psi|^4, \tag{8}$$

where $B > 0$. When the coefficient A of the *quadratic term* $|\psi|^2$ is negative $A < 0$, this free energy has a minimum at a finite value of $|\psi|$, ($|\psi| \neq 0$). On the other hand, when $A \geq 0$, it has a minimum at $|\psi| = 0$.

Keeping the above in mind, we consider the *quadratic terms* ($\equiv f_{y2}$) of η_y in the Ginzburg-Landau free energy density [Eq. (7)],

$$f = \Big\{a'(T - T_{cx})|\eta_x|^2 + b_1|\eta_x|^4\Big\} + f_{y2} + b_1|\eta_y|^4, \tag{9}$$

where the quadratic terms of η_y are

$$f_{y2} = a'(T - T_{cy})|\eta_y|^2 + 2b_1|\eta_x|^2|\eta_y|^2 + \frac{b_2}{2}\{\eta_x^{*2}\eta_y^2 + \eta_x^2\eta_y^{*2}\} + b_3|\eta_x|^2|\eta_y|^2 \tag{10}$$

$$= \Big[a'(T - T_{cy}) + (2b_1 + b_3)|\eta_x|^2\Big]|\eta_y|^2 + \frac{b_2}{2}\{\eta_x^{*2}\eta_y^2 + \eta_x^2\eta_y^{*2}\}. \tag{11}$$

f_{y2} can be rewritten as

$$f_{y2} = (\,\eta_y^*,\quad \eta_y\,)\,\frac{1}{2}\begin{pmatrix} a'(T - T_{cy}) + (2b_1 + b_3)|\eta_x|^2 & b_2\eta_x^2 \\ b_2\eta_x^{*2} & a'(T - T_{cy}) + (2b_1 + b_3)|\eta_x|^2 \end{pmatrix}\begin{pmatrix} \eta_y \\ \eta_y^* \end{pmatrix} \tag{12}$$

$$\equiv \frac{1}{2}(\,\eta_y^*,\quad \eta_y\,)\,\hat{A}\begin{pmatrix} \eta_y \\ \eta_y^* \end{pmatrix}. \tag{13}$$

Now,

$$\hat{A} \equiv \begin{pmatrix} a & b \\ c & a \end{pmatrix}. \tag{14}$$

$$\det(\hat{A} - E\hat{1}) = 0. \tag{15}$$

$$\rightarrow \qquad (a-E)^2 - bc = 0. \tag{16}$$

$$\rightarrow \qquad E^2 - 2aE + a^2 - bc = 0. \tag{17}$$

$$\begin{aligned}
\rightarrow \qquad E &= a \pm \sqrt{a^2 - (a^2 - bc)} & (18)\\
&= a \pm \sqrt{bc} & (19)\\
&= a'(T - T_{cy}) + (2b_1 + b_3)|\eta_x|^2 \pm \sqrt{(b_2\eta_x^2)(b_2\eta_x^{*2})} & (20)\\
&= a'(T - T_{cy}) + (2b_1 + b_3)|\eta_x|^2 \pm \sqrt{b_2^2|\eta_x|^4} & (21)\\
&= a'(T - T_{cy}) + (2b_1 + b_3)|\eta_x|^2 \pm b_2|\eta_x|^2 & (22)\\
&= a'(T - T_{cy}) + (2b_1 + b_3 \pm b_2)|\eta_x|^2. & (23)
\end{aligned}$$

The eigen vectors are

$$\begin{pmatrix} b \\ \sqrt{bc} \end{pmatrix} \qquad \text{for} \quad E = a + \sqrt{bc} \quad \Big(E = a'(T - T_{cy}) + (2b_1 + b_3 + b_2)|\eta_x|^2\Big), \tag{24}$$

$$\begin{pmatrix} b \\ -\sqrt{bc} \end{pmatrix} \qquad \text{for} \quad E = a - \sqrt{bc} \quad \Big(E = a'(T - T_{cy}) + (2b_1 + b_3 - b_2)|\eta_x|^2\Big). \tag{25}$$

From these eigen vectors, we define the following matrices ($\hat{U}\hat{U}^{-1} = \hat{U}^{-1}\hat{U} = \hat{1}$),

$$\hat{U} = \begin{pmatrix} b & b \\ \sqrt{bc} & -\sqrt{bc} \end{pmatrix}, \qquad \hat{U}^{-1} = \frac{1}{2b\sqrt{bc}} \begin{pmatrix} \sqrt{bc} & b \\ \sqrt{bc} & -b \end{pmatrix}. \tag{26}$$

The matrix $\hat{A}$ is then digonalized as

$$\begin{aligned}
\hat{U}^{-1}\hat{A}\hat{U} &= \frac{1}{2b\sqrt{bc}} \begin{pmatrix} \sqrt{bc} & b \\ \sqrt{bc} & -b \end{pmatrix} \begin{pmatrix} a & b \\ c & a \end{pmatrix} \begin{pmatrix} b & b \\ \sqrt{bc} & -\sqrt{bc} \end{pmatrix} & (27)\\
&= \frac{1}{2b\sqrt{bc}} \begin{pmatrix} \sqrt{bc} & b \\ \sqrt{bc} & -b \end{pmatrix} \begin{pmatrix} ab + b\sqrt{bc} & ab - b\sqrt{bc} \\ bc + a\sqrt{bc} & bc - a\sqrt{bc} \end{pmatrix} & (28)\\
&= \frac{1}{2b\sqrt{bc}} \begin{pmatrix} \sqrt{bc} & b \\ \sqrt{bc} & -b \end{pmatrix} \begin{pmatrix} b(a + \sqrt{bc}) & b(a - \sqrt{bc}) \\ \sqrt{bc}(a + \sqrt{bc}) & -\sqrt{bc}(a - \sqrt{bc}) \end{pmatrix} & (29)\\
&= \frac{1}{2b\sqrt{bc}} \begin{pmatrix} 2b\sqrt{bc}(a + \sqrt{bc}) & (b\sqrt{bc} - b\sqrt{bc})(a - \sqrt{bc}) \\ (b\sqrt{bc} + b\sqrt{bc})(a - \sqrt{bc}) & 2b\sqrt{bc}(a - \sqrt{bc}) \end{pmatrix} & (30)\\
&= \begin{pmatrix} a + \sqrt{bc} & 0 \\ 0 & a - \sqrt{bc} \end{pmatrix}. & (31)
\end{aligned}$$

And

$$\begin{aligned}
\hat{U}^{-1} \begin{pmatrix} \eta_y \\ \eta_y^* \end{pmatrix} &= \frac{1}{2b\sqrt{bc}} \begin{pmatrix} \sqrt{bc} & b \\ \sqrt{bc} & -b \end{pmatrix} \begin{pmatrix} \eta_y \\ \eta_y^* \end{pmatrix} & (32)\\
&= \frac{1}{2b\sqrt{bc}} \begin{pmatrix} \sqrt{bc}\eta_y + b\eta_y^* \\ \sqrt{bc}\eta_y - b\eta_y^* \end{pmatrix} & (33)
\end{aligned}$$

$$\begin{aligned}
(\,\eta_y^*, \quad \eta_y\,)\,\hat{U} &= (\,\eta_y^*, \quad \eta_y\,) \begin{pmatrix} b & b \\ \sqrt{bc} & -\sqrt{bc} \end{pmatrix} & (34)\\
&= (\,\sqrt{bc}\eta_y + b\eta_y^*, \quad -(\sqrt{bc}\eta_y - b\eta_y^*)\,) & (35)
\end{aligned}$$

Hence, f_{y2} is expressed as

$$f_{y2} = \frac{1}{2} \begin{pmatrix} \eta_y^*, & \eta_y \end{pmatrix} \hat{A} \begin{pmatrix} \eta_y \\ \eta_y^* \end{pmatrix} \tag{36}$$

$$= \frac{1}{2} \begin{pmatrix} \eta_y^*, & \eta_y \end{pmatrix} \hat{U}(\hat{U}^{-1}\hat{A}\hat{U})\hat{U}^{-1} \begin{pmatrix} \eta_y \\ \eta_y^* \end{pmatrix} \tag{37}$$

$$= \begin{pmatrix} \sqrt{bc}\eta_y + b\eta_y^*, & -(\sqrt{bc}\eta_y - b\eta_y^*) \end{pmatrix} \begin{pmatrix} a+\sqrt{bc} & 0 \\ 0 & a-\sqrt{bc} \end{pmatrix} \frac{1}{2b\sqrt{bc}} \begin{pmatrix} \sqrt{bc}\eta_y + b\eta_y^* \\ \sqrt{bc}\eta_y - b\eta_y^* \end{pmatrix} \tag{38}$$

$$= \frac{1}{2b\sqrt{bc}} \Big[(a+\sqrt{bc})(\sqrt{bc}\eta_y + b\eta_y^*)^2 - (a-\sqrt{bc})(\sqrt{bc}\eta_y - b\eta_y^*)^2 \Big] \tag{39}$$

Here, without loss of generality, we assume η_x is real. Then, recalling the definitions $b = b_2\eta_x^2$ and $c = b_2\eta_x^{*2}$, we have a relation $\sqrt{bc} = b = b_2\eta_x^2$. Thus,

$$f_{y2} = \frac{1}{2b\sqrt{bc}} \Big[(a+\sqrt{bc})(\sqrt{bc}\eta_y + b\eta_y^*)^2 - (a-\sqrt{bc})(\sqrt{bc}\eta_y - b\eta_y^*)^2 \Big] \tag{40}$$

$$= \frac{1}{2b^2} \Big[(a+b)(b\eta_y + b\eta_y^*)^2 - (a-b)(b\eta_y - b\eta_y^*)^2 \Big] \tag{41}$$

$$= \frac{1}{2} \Big[(a+b)(\eta_y + \eta_y^*)^2 - (a-b)(\eta_y - \eta_y^*)^2 \Big]. \tag{42}$$

The order parameters $\eta_y + \eta_y^*$ and $\eta_y - \eta_y^*$ are real and imaginary numbers, respectively. We, therefore, define $\eta_{yr} \equiv \eta_y + \eta_y^*$ and $i\eta_{yi} \equiv \eta_y - \eta_y^*$ with η_{yr} and η_{yi} real. Then,

$$f_{y2} = \frac{1}{2} \Big[(a+b)(\eta_{yr})^2 - (a-b)(i\eta_{yi})^2 \Big] \tag{43}$$

$$= \frac{1}{2} \Big[(a+b)\eta_{yr}^2 + (a-b)\eta_{yi}^2 \Big]. \tag{44}$$

Because we have assumed η_x is real, the coefficients $a \pm b$ are

$$a \pm b = a'(T - T_{\rm cy}) + (2b_1 + b_3)|\eta_x|^2 \pm b_2\eta_x^2 \tag{45}$$

$$= a'(T - T_{\rm cy}) + (2b_1 + b_3 \pm b_2)|\eta_x|^2 \tag{46}$$

$$= a' \left(T - T_{\rm cy} + \frac{(2b_1 + b_3 \pm b_2)|\eta_x|^2}{a'} \right) \tag{47}$$

$$= a' \Big(T - T_\pm \Big), \tag{48}$$

where

$$T_\pm \equiv T_{cy} - \frac{(2b_1 + b_3 \pm b_2)|\eta_x|^2}{a'}. \tag{49}$$

Substituting $|\eta_x(T)|^2$ in Eq. (6) into this,

$$T_\pm \equiv T_{cy} - \frac{(2b_1 + b_3 \pm b_2)\frac{-a'(T - T_{\rm cx})}{2b_1}}{a'} \tag{50}$$

$$= T_{cy} + \frac{(2b_1 + b_3 \pm b_2)(T - T_{\rm cx})}{2b_1} \tag{51}$$

$$\equiv T_{cy} + R_\pm(T - T_{\rm cx}). \tag{52}$$

Therefore,

$$\begin{aligned} a \pm b &= a'\Big(T - T_{\pm}\Big), && (53)\\ &= a'T - a'\Big[T_{cy} + R_{\pm}(T - T_{\rm cx})\Big] && (54)\\ &= a'(1 - R_{\pm})T - a'T_{cy} + a'R_{\pm}T_{cx} && (55)\\ &= a'(1 - R_{\pm})\left(T - \frac{T_{\rm cy} - R_{\pm}T_{\rm cx}}{1 - R_{\pm}}\right) && (56)\\ &= a'(1 - R_{\pm})\Big(T - T_{y\pm}\Big). && (57) \end{aligned}$$

Here,

$$\begin{aligned} T_{y\pm} &\equiv \frac{T_{\rm cy} - R_{\pm}T_{\rm cx}}{1 - R_{\pm}} && (58)\\ &= T_{\rm cy}\frac{1 - R_{\pm}T_{\rm cx}/T_{\rm cy}}{1 - R_{\pm}}, && (59) \end{aligned}$$

with

$$\begin{aligned} R_{\pm} &= \frac{2b_1 + b_3 \pm b_2}{2b_1} && (60)\\ &= 1 + \frac{b_3 \pm b_2}{2b_1}. && (61) \end{aligned}$$

Finally, from Eqs. (9), (44), (57), the Ginzburg-Landau free energy density is expressed as follows.

$$\begin{aligned} f &= \Big\{a'(T - T_{\rm cx})|\eta_x|^2 + b_1|\eta_x|^4\Big\} + f_{y2} + b_1|\eta_y|^4 && (62)\\ &= \Big\{a'(T - T_{\rm cx})|\eta_x|^2 + b_1|\eta_x|^4\Big\} + \frac{1}{2}\Big[(a+b)\eta_{yr}^2 + (a-b)\eta_{yi}^2\Big] + b_1|\eta_y|^4 && (63)\\ &= \Big\{a'(T - T_{\rm cx})|\eta_x|^2 + b_1|\eta_x|^4\Big\} + \frac{1}{2}\Big[a'(1 - R_+)\Big(T - T_{y+}\Big)\eta_{yr}^2 + a'(1 - R_-)\Big(T - T_{y-}\Big)\eta_{yi}^2\Big] + b_1|\eta_y|^4. && (64) \end{aligned}$$

From the coefficients of the quadratic terms of η_y, we conclude that the second transition temperature $T'_{\rm cy}$ is given by

$$T'_{\rm cy} = \max\{T_{y+}, T_{y-}\}. \qquad (65)$$

At $T < T'_{\rm cy}$, $\vec{\eta} = (\eta_x, \eta_{yr})$ if $T'_{\rm cy} = T_{y+}$. On the other hand, $\vec{\eta} = (\eta_x, i\eta_{yi})$ if $T'_{\rm cy} = T_{y-}$. (Here, η_x, η_{yr}, and η_{yi} are assumed to be real.)