Unkonventionelle Supraleitung

Serie 10

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The condensation energy near the critical temperature T_c.

10.1 Let us consider the unitary triplet states $\hat{\Delta}_k = \vec{d}_k \cdot \hat{\vec{\sigma}} i\hat{\sigma}_y$, where $\vec{d}_k \times \vec{d}_k^* = 0$ and $|\Delta_k| = \sqrt{\frac{1}{2}\mathrm{Tr}[\hat{\Delta}_k\hat{\Delta}_k^\dagger]} = |\vec{d}_k|$. We define the amplitude $|\eta| \equiv \sqrt{\left\langle |\vec{d}_k|^2 \right\rangle_k}$ and the normalized d vector $\hat{d}_k \equiv \vec{d}_k/|\eta|$. Thus, $\left\langle |\hat{d}_k|^2 \right\rangle_k = 1$. Here, $\langle \cdots \rangle_k$ denotes the average over the spherical Fermi surface, $\langle \cdots \rangle_k = \int \frac{d\Omega_k}{4\pi} \cdots$.

The Ginzburg-Landau free energy for spatially uniform systems is given as follows *within the weak-coupling BCS theory.*

$$\begin{aligned} F &= F_n - \alpha\Big(1 - \frac{T}{T_c}\Big)\left\langle |\Delta_k|^2 \right\rangle_k + \frac{\beta}{2}\left\langle |\Delta_k|^4 \right\rangle_k \\ &= F_n - \alpha\Big(1 - \frac{T}{T_c}\Big)\left\langle |\vec{d}_k|^2 \right\rangle_k + \frac{\beta}{2}\left\langle |\vec{d}_k|^4 \right\rangle_k \\ &= F_n - \alpha\Big(1 - \frac{T}{T_c}\Big)\left\langle |\hat{d}_k|^2 \right\rangle_k |\eta|^2 + \frac{\beta}{2}\left\langle |\hat{d}_k|^4 \right\rangle_k |\eta|^4 \\ &= F_n - \alpha\Big(1 - \frac{T}{T_c}\Big)|\eta|^2 + \frac{\beta}{2}\left\langle |\hat{d}_k|^4 \right\rangle_k |\eta|^4, \end{aligned}$$

where $\alpha = N_0$ (> 0) and $\beta = 7\zeta(3)N_0/8(\pi k_B T_c)^2$ (> 0). F_n and N_0 are the free energy and the density of states (per spin projection) in the normal state, respectively. $\zeta(3) = \sum_{n=1}^{\infty} n^{-3} \approx 1.2$ is the Riemann zeta function.

a) Show that below T_c, the above free energy has a minimum value with respect to η ($\partial F/\partial \eta^* = 0$), when η is

$$|\eta|^2 = \frac{\alpha}{\beta\left\langle |\hat{d}_k|^4 \right\rangle_k}\Big(1 - \frac{T}{T_c}\Big).$$

Show also that for this $|\eta|^2$, the condensation energy $F_{cond} = F - F_n$ is given as

$$F_{cond} = \frac{-\alpha^2}{2\beta\left\langle |\hat{d}_k|^4 \right\rangle_k}\Big(1 - \frac{T}{T_c}\Big)^2. \tag{1}$$

On the other hand, when taking into account the so-called *feedback effect* in the superfluid ^{3}He, the following term is added to the free energy according to the Leggett's review article.[1]

$$F_{FB} = \frac{\beta}{2}\gamma|\eta|^4 \sum_{i,j}\Big[\mathrm{Re}\left\langle \hat{d}_k^{i*}\hat{d}_k^j \right\rangle_k \Big(\delta_{i,j} - 2\mathrm{Re}\left\langle \hat{d}_k^{i*}\hat{d}_k^j \right\rangle_k\Big)\Big]$$

[1] A. J. Leggett, Rev. Mod. Phys. **47**, 331 (1975). See Eq. (9.7) therein.

$$
\begin{aligned}
&= \frac{\beta}{2}\gamma|\eta|^4\Big[\Big\langle|\hat{d}_k|^2\Big\rangle_k - 2\Big\{\Big\langle|\hat{d}_k^x|^2\Big\rangle_k\Big\}^2 - 2\Big\{\Big\langle|\hat{d}_k^y|^2\Big\rangle_k\Big\}^2 - 2\Big\{\Big\langle|\hat{d}_k^z|^2\Big\rangle_k\Big\}^2 \\
&\qquad - 2\Big\{\mathrm{Re}\Big\langle\hat{d}_k^{x*}\hat{d}_k^y\Big\rangle_k\Big\}^2 - 2\Big\{\mathrm{Re}\Big\langle\hat{d}_k^{y*}\hat{d}_k^x\Big\rangle_k\Big\}^2 \\
&\qquad - 2\Big\{\mathrm{Re}\Big\langle\hat{d}_k^{y*}\hat{d}_k^z\Big\rangle_k\Big\}^2 - 2\Big\{\mathrm{Re}\Big\langle\hat{d}_k^{z*}\hat{d}_k^y\Big\rangle_k\Big\}^2 \\
&\qquad - 2\Big\{\mathrm{Re}\Big\langle\hat{d}_k^{z*}\hat{d}_k^x\Big\rangle_k\Big\}^2 - 2\Big\{\mathrm{Re}\Big\langle\hat{d}_k^{x*}\hat{d}_k^z\Big\rangle_k\Big\}^2\Big] \\
&= \frac{\beta}{2}\gamma|\eta|^4\Big[1 - 2\Big\{\Big\langle|\hat{d}_k^x|^2\Big\rangle_k\Big\}^2 - 2\Big\{\Big\langle|\hat{d}_k^y|^2\Big\rangle_k\Big\}^2 - 2\Big\{\Big\langle|\hat{d}_k^z|^2\Big\rangle_k\Big\}^2 \\
&\qquad - 2\Big\{\frac{1}{2}\Big\langle\hat{d}_k^{x*}\hat{d}_k^y + \hat{d}_k^{y*}\hat{d}_k^x\Big\rangle_k\Big\}^2 - 2\Big\{\frac{1}{2}\Big\langle\hat{d}_k^{y*}\hat{d}_k^x + \hat{d}_k^{x*}\hat{d}_k^y\Big\rangle_k\Big\}^2 \\
&\qquad - 2\Big\{\frac{1}{2}\Big\langle\hat{d}_k^{y*}\hat{d}_k^z + \hat{d}_k^{z*}\hat{d}_k^y\Big\rangle_k\Big\}^2 - 2\Big\{\frac{1}{2}\Big\langle\hat{d}_k^{z*}\hat{d}_k^y + \hat{d}_k^{y*}\hat{d}_k^z\Big\rangle_k\Big\}^2 \\
&\qquad - 2\Big\{\frac{1}{2}\Big\langle\hat{d}_k^{z*}\hat{d}_k^x + \hat{d}_k^{x*}\hat{d}_k^z\Big\rangle_k\Big\}^2 - 2\Big\{\frac{1}{2}\Big\langle\hat{d}_k^{x*}\hat{d}_k^z + \hat{d}_k^{z*}\hat{d}_k^x\Big\rangle_k\Big\}^2\Big], \qquad \Big(\Big\langle|\hat{d}_k|^2\Big\rangle_k = 1\Big) \\
&= \frac{\beta}{2}\gamma|\eta|^4\Big[1 - 2\Big\{\Big\langle|\hat{d}_k^x|^2\Big\rangle_k\Big\}^2 - 2\Big\{\Big\langle|\hat{d}_k^y|^2\Big\rangle_k\Big\}^2 - 2\Big\{\Big\langle|\hat{d}_k^z|^2\Big\rangle_k\Big\}^2 \\
&\qquad - 4\Big\{\frac{1}{2}\Big\langle\hat{d}_k^{x*}\hat{d}_k^y + \hat{d}_k^{y*}\hat{d}_k^x\Big\rangle_k\Big\}^2 \\
&\qquad - 4\Big\{\frac{1}{2}\Big\langle\hat{d}_k^{y*}\hat{d}_k^z + \hat{d}_k^{z*}\hat{d}_k^y\Big\rangle_k\Big\}^2 \\
&\qquad - 4\Big\{\frac{1}{2}\Big\langle\hat{d}_k^{z*}\hat{d}_k^x + \hat{d}_k^{x*}\hat{d}_k^z\Big\rangle_k\Big\}^2\Big] \\
&= \frac{\beta}{2}|\eta|^4\Big[\gamma - 2\gamma\Big\{\Big\langle|\hat{d}_k^x|^2\Big\rangle_k\Big\}^2 - 2\gamma\Big\{\Big\langle|\hat{d}_k^y|^2\Big\rangle_k\Big\}^2 - 2\gamma\Big\{\Big\langle|\hat{d}_k^z|^2\Big\rangle_k\Big\}^2 \\
&\qquad\quad - 4\gamma\Big\{\Big\langle\hat{d}_k^{x*}\hat{d}_k^y\Big\rangle_k\Big\}^2 - 4\gamma\Big\{\Big\langle\hat{d}_k^{y*}\hat{d}_k^z\Big\rangle_k\Big\}^2 - 4\gamma\Big\{\Big\langle\hat{d}_k^{z*}\hat{d}_k^x\Big\rangle_k\Big\}^2\Big]. \qquad (\hat{d}_k^* \times \hat{d}_k = 0)
\end{aligned}
$$

Here, γ is the numeric factor which represents the strength of the feedback effect. Then, the Ginzburg-Landau free energy is given as

$$
\begin{aligned}
F &= F_n - \alpha\Big(1 - \frac{T}{T_\mathrm{c}}\Big)|\eta|^2 + \frac{\beta}{2}\Big\langle|\hat{d}_k|^4\Big\rangle_k|\eta|^4 + F_{FB} \\
&= F_n - \alpha\Big(1 - \frac{T}{T_\mathrm{c}}\Big)|\eta|^2 + \frac{\beta}{2}\kappa|\eta|^4,
\end{aligned}
$$

where

$$
\begin{aligned}
\kappa &= \Big\langle|\hat{d}_k|^4\Big\rangle_k + \gamma - 2\gamma\Big\{\Big\langle|\hat{d}_k^x|^2\Big\rangle_k\Big\}^2 - 2\gamma\Big\{\Big\langle|\hat{d}_k^y|^2\Big\rangle_k\Big\}^2 - 2\gamma\Big\{\Big\langle|\hat{d}_k^z|^2\Big\rangle_k\Big\}^2 \\
&\quad - 4\gamma\Big\{\Big\langle\hat{d}_k^{x*}\hat{d}_k^y\Big\rangle_k\Big\}^2 - 4\gamma\Big\{\Big\langle\hat{d}_k^{y*}\hat{d}_k^z\Big\rangle_k\Big\}^2 - 4\gamma\Big\{\Big\langle\hat{d}_k^{z*}\hat{d}_k^x\Big\rangle_k\Big\}^2.
\end{aligned}
$$

b) Show that below T_c, this free energy has a minimum at

$$
|\eta|^2 = \frac{\alpha}{\beta\kappa}\Big(1 - \frac{T}{T_\mathrm{c}}\Big).
$$

Show also that for this $|\eta|^2$, the condensation energy $F_{cond} = F - F_n$ is given as

$$
F_{cond} = \frac{-\alpha^2}{2\beta\kappa}\Big(1 - \frac{T}{T_\mathrm{c}}\Big)^2. \tag{2}
$$

10.2 Now, let us compare the condensation energies in the A- and B-phases of the superfluid ^{3}He on the basis of Eqs. (1) and (2). Assume the following normalized d vectors: the A-phase $\hat{d}_k = \sqrt{\frac{3}{2}}(0, 0, \hat{k}_x + i\hat{k}_y)$, and the B-phase $\hat{d}_k = (\hat{k}_x, \hat{k}_y, \hat{k}_z)$. Here, $\hat{k} = (\hat{k}_x, \hat{k}_y, \hat{k}_z) \equiv \vec{k}/|\vec{k}|$.

a) Show that the ratio of the free energies in the A- and B-phases is given as follows, *within the weak-coupling BCS theory* (Eq. (1)).

$$\frac{F_{cond}^{(A)}}{F_{cond}^{(B)}} = \frac{5}{6}.$$

(This result indicates that $|F_{cond}^{(B)}| > |F_{cond}^{(A)}|$, namely that the B phase is energetically more favorable than the A phase within the weak-coupling BCS theory.)

b) Show that the ratio of the free energies in the A- and B-phases is given as follows, *in the case when the feedback effect is taken into account* (Eq. (2)).

$$\frac{F_{cond}^{(A)}}{F_{cond}^{(B)}} = \frac{1 + \frac{1}{3}\gamma}{\frac{6}{5} - \gamma}.$$

Also, evaluate the region of γ in which $|F_{cond}^{(A)}| > |F_{cond}^{(B)}|$, namely in which the A-phase is energetically more favorable than the B-phase.
(Here, assume that γ is a small positive quantity, and then $0 \le \gamma < \frac{6}{5}$.)

Hint: $(\hat{k}_x, \hat{k}_y, \hat{k}_z) = (\cos\phi\sin\theta,\ \sin\phi\sin\theta,\ \cos\theta)$.

$$\int_0^{2\pi} d\phi \cos^2\phi = \int_0^{2\pi} d\phi \sin^2\phi = \pi,$$

$$\int_0^{\pi} d\theta \sin^3\theta = \frac{4}{3}, \qquad \int_0^{\pi} d\theta \sin\theta\cos^2\theta = \frac{2}{3}.$$

$$\int_0^{\pi} d\theta \sin^5\theta = \frac{16}{15},$$

$$\int \frac{d\Omega_k}{4\pi} \cdots = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \cdots.$$