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11 Let us consider the superconductivity in a system with tetragonal crystal structure and strong spin-orbit coupling discussed in the theory lecture notes. For superconducting phases described by two-component order parameter $\vec{\eta} = (\eta_x, \eta_y)$ corresponding to two basis functions of the two dimensional representation [Eq. (4.20) or (166) in the theory lecture notes], the Ginzburg-Landau free energy density f *in the spatially uniform case under zero magnetic field* is given as follows [see Eq. (4.21) or (167)].

$$
\begin{aligned}
f &= a(T)|\vec{\eta}|^2 + b_1|\vec{\eta}|^4 + \frac{b_2}{2}\{\eta_x^{*2}\eta_y^2 + \eta_x^2\eta_y^{*2}\} + b_3|\eta_x|^2|\eta_y|^2 &(1)\\
&= a(T)\{|\eta_x|^2 + |\eta_y|^2\} + b_1\Big|\{|\eta_x|^2 + |\eta_y|^2\}\Big|^2 + \frac{b_2}{2}\{\eta_x^{*2}\eta_y^2 + \eta_x^2\eta_y^{*2}\} + b_3|\eta_x|^2|\eta_y|^2 &(2)\\
&= a(T)\{|\eta_x|^2 + |\eta_y|^2\} + b_1\Big\{|\eta_x|^4 + |\eta_y|^4 + 2|\eta_x|^2|\eta_y|^2\Big\} + \frac{b_2}{2}\{\eta_x^{*2}\eta_y^2 + \eta_x^2\eta_y^{*2}\} + b_3|\eta_x|^2|\eta_y|^2. &(3)
\end{aligned}
$$

Here, we have omitted the gradient terms and the magnetic field term because of the spatially uniformity assumed. The coefficients $a(T)$ and b_i $(i = 1, 2, 3)$ are real numbers. $a(T) < 0$ for $T < T_c$.

a) Taking the variation of f with respect to η_x^* $(\partial f/\partial\eta_x^* = 0)$ and η_y^* $(\partial f/\partial\eta_y^* = 0)$, we obtain two coupled Ginzburg-Landau equations:

$$
\begin{aligned}
a(T)\eta_x + b_1\Big\{2|\eta_x|^2\eta_x + 2\eta_x|\eta_y|^2\Big\} + b_2\eta_x^*\eta_y^2 + b_3\eta_x|\eta_y|^2 &= 0, &(4)\\
a(T)\eta_y + b_1\Big\{2|\eta_y|^2\eta_y + 2|\eta_x|^2\eta_y\Big\} + b_2\eta_x^2\eta_y^* + b_3|\eta_x|^2\eta_y &= 0. &(5)
\end{aligned}
$$

$$
\begin{aligned}
\rightarrow \qquad a(T)\eta_x + 2b_1\Big\{|\eta_x|^2 + |\eta_y|^2\Big\}\eta_x + b_2\eta_y^2\eta_x^* + b_3|\eta_y|^2\eta_x &= 0, &(6)\\
a(T)\eta_y + 2b_1\Big\{|\eta_y|^2 + |\eta_x|^2\Big\}\eta_y + b_2\eta_x^2\eta_y^* + b_3|\eta_x|^2\eta_y &= 0. &(7)
\end{aligned}
$$

b) Let us parameterize the order parameters η_x and η_y as

$$
(\eta_x, \eta_y) = (\eta_0\cos\alpha,\ \eta_0 e^{i\gamma}\sin\alpha), \qquad (8)
$$

with η_0 real, α $(-\pi/2 < \alpha \leq \pi/2)$, and γ $(0 \leq \gamma < 2\pi)$.

Substituting this into the Ginzburg-Landau free energy density (2),

$$
\begin{aligned}
f &= a(T)\{|\eta_x|^2 + |\eta_y|^2\} + b_1\Big|\{|\eta_x|^2 + |\eta_y|^2\}\Big|^2 + \frac{b_2}{2}\{\eta_x^{*2}\eta_y^2 + \eta_x^2\eta_y^{*2}\} + b_3|\eta_x|^2|\eta_y|^2 &(9)\\
&= a(T)\{\cos^2\alpha + \sin^2\alpha\}\eta_0^2 + b_1\Big|\{\cos^2\alpha + \sin^2\alpha\}\Big|^2\eta_0^4 \\
&\quad + \frac{b_2}{2}\{e^{2i\gamma}\cos^2\alpha\sin^2\alpha + e^{-2i\gamma}\cos^2\alpha\sin^2\alpha\}\eta_0^4 + b_3(\cos^2\alpha\sin^2\alpha)\eta_0^4 &(10)
\end{aligned}
$$

$$= a(T)\eta_0^2 + b_1\eta_0^4 + b_2\{\cos 2\gamma(\cos\alpha\sin\alpha)^2\}\eta_0^4 + b_3(\cos\alpha\sin\alpha)^2\eta_0^4 \tag{11}$$

$$= a(T)\eta_0^2 + b_1\eta_0^4 + \frac{b_2}{2^2}\{\cos 2\gamma(\sin 2\alpha)^2\}\eta_0^4 + \frac{b_3}{2^2}(\sin 2\alpha)^2\eta_0^4 \tag{12}$$

$$= a(T)\eta_0^2 + \frac{1}{4}\Big[4b_1 + \sin^2 2\alpha(b_2\cos 2\gamma + b_3)\Big]\eta_0^4. \tag{13}$$

From now on, let us assume $b_1 > 0$.

c) Because $a(T) < 0$, the condition for f to have a minimum with respect to η_0 is that the coefficient of the second term in Eq. (13) is positive, namely

$$4b_1 + \sin^2 2\alpha(b_2\cos 2\gamma + b_3) > 0. \tag{14}$$

We define $B \equiv 4b_1 + \sin^2 2\alpha(b_2\cos 2\gamma + b_3)$. The free energy density f [Eq. (13)] reads:

$$f = a(T)\eta_0^2 + \frac{1}{4}B\eta_0^4. \tag{15}$$

Taking the variation of f with respect to η_0 $(\partial f/\partial\eta_0 = 0)$,

$$2a(T)\eta_0 + B\eta_0^3 = 0. \tag{16}$$

$$\rightarrow \qquad 2a(T) + B\eta_0^2 = 0. \tag{17}$$

$$\rightarrow \qquad \eta_0^2 = \frac{-2a(T)}{B} \tag{18}$$

$$= \frac{-2a(T)}{4b_1 + \sin^2 2\alpha(b_2\cos 2\gamma + b_3)}. \tag{19}$$

This η_0^2 yields a minimum of f. Substituting it into Eq. (15), we obtain the minimum of f as

$$f = a(T)\left(\frac{-2a(T)}{B}\right) + \frac{1}{4}B\left(\frac{-2a(T)}{B}\right)^2 \tag{20}$$

$$= \frac{-2\{a(T)\}^2}{B} + \frac{1}{4}B\left(\frac{4\{a(T)\}^2}{B^2}\right) \tag{21}$$

$$= \frac{-2\{a(T)\}^2}{B} + \frac{\{a(T)\}^2}{B} \tag{22}$$

$$= \frac{-\{a(T)\}^2}{B} \tag{23}$$

$$= \frac{-\{a(T)\}^2}{4b_1 + \sin^2 2\alpha(b_2\cos 2\gamma + b_3)}. \tag{24}$$

d) Next, let us minimize f with respect to the parameter α. The free energy density f in Eq. (24) has a minimum when its denominator (> 0) is minimized.

If $(b_2 \cos 2\gamma + b_3) > 0$, the free energy density in Eq. (24) has a minimum when $\sin^2 2\alpha = 0$, namely when $\alpha = 0,\ \pi/2$. On the other hand, if $(b_2 \cos 2\gamma + b_3) < 0$, the free energy density has a minimum when $\sin^2 2\alpha = 1$, namely when $\alpha = \pm\pi/4$. Recalling $(\eta_x, \eta_y) = (\eta_0 \cos\alpha,\ \eta_0 e^{i\gamma} \sin\alpha)$, we have

$$(\eta_x, \eta_y) = \begin{cases} \left(\frac{\eta_0}{\sqrt{2}},\ \pm\frac{\eta_0}{\sqrt{2}} e^{i\gamma}\right) & (b_2 \cos 2\gamma + b_3 < 0, \quad 4b_1 + b_2 \cos 2\gamma + b_3 > 0) \\ (\eta_0, 0) \quad \text{or} \quad (0, \eta_0 e^{i\gamma}) & (b_2 \cos 2\gamma + b_3 > 0, \quad 4b_1 > 0) \end{cases} \tag{25}$$

Here, we have taken account of the condition represented by Eq. (14), too.

e) Finally, let us minimize f with respect to the parameter γ.

When $(b_2 \cos 2\gamma + b_3) < 0$, the smallest value of $b_2 \cos 2\gamma$ minimizes f. In this case, the values $\gamma = 0$ and $\gamma = \pi$ minimize f if $b_2 < 0$, while $\gamma = \pi/2$ and $\gamma = 3\pi/2$ minimize f if $b_2 > 0$. That is to say,

$$(\eta_x, \eta_y) = \begin{cases} \left(\frac{\eta_0}{\sqrt{2}},\ \pm i\frac{\eta_0}{\sqrt{2}}\right) & (-b_2 + b_3 < 0, \quad b_2 > 0, \quad 4b_1 - b_2 + b_3 > 0) \\ \left(\frac{\eta_0}{\sqrt{2}},\ \pm\frac{\eta_0}{\sqrt{2}}\right) & (b_2 + b_3 < 0, \quad b_2 < 0, \quad 4b_1 + b_2 + b_3 > 0) \\ (\eta_0, 0) \quad \text{or} \quad (0, \eta_0 e^{i\gamma}) & (b_2 \cos 2\gamma + b_3 > 0, \quad b_1 > 0) \end{cases} \tag{26}$$

When $(b_2 \cos 2\gamma + b_3) > 0$, $\sin^2 2\alpha = 0$ as noticed above. In this case, the free energy density f in Eq. (24) is independent of γ. Therefore, we can choose the phase γ arbitrarily.

The superconducting phases represented by the order parameters in Eq. (26) correspond to the phases A, B, and C discussed in the theory lecture notes (see Fig. 7.1 or Fig. 10 therein). The phases A, B, and C correspond to the order parameters from top to bottom in Eq. (26), respectively. From the above result, one can easily confirm that the free energy density ($f < 0$) in Eq. (24) is lower for the phases A and B [corresponding to the first (A) and second (B) lines in Eq. (26)] than for the phase C [the third line]. (Consider the denominator of f, the value of α, and the sign of the factor $(b_2 \cos 2\gamma + b_3)$ for each phase.) Therefore, concerning the phase diagram in the (b_2, b_3) parameter space, the phase A or B is energetically more favorable than the C phase in the region where the inequalities in the first or second line in Eq. (26) are satisfied.