Unkonventionelle Supraleitung WS 05/06
Lösungen zur Serie 1

1.1 First of all, the compressibility κ is defined by

$$\kappa^{-1} = -V\frac{\partial p}{\partial V}. \tag{1}$$

The pressure p is

$$p = -\frac{\partial E}{\partial V} \qquad \rightarrow \qquad \frac{\partial p}{\partial V} = -\frac{\partial^2 E}{\partial V^2}. \tag{2}$$

From these,

$$\kappa^{-1} = V\frac{\partial^2 E}{\partial V^2}. \tag{3}$$

Now, let us assume:

$$E = Vf(\rho) \qquad \text{with} \qquad \rho = \frac{N}{V}. \tag{4}$$

Then,

$$\frac{\partial E}{\partial V} = f(\rho) + Vf'(\rho)\frac{\partial \rho}{\partial V}. \tag{5}$$

Noting ρ is a function of the two variables N and V (i.e., $\rho = N/V$),

$$\left.\frac{\partial \rho}{\partial V}\right|_N = -\frac{N}{V^2} \qquad \text{and} \qquad \left.\frac{\partial \rho}{\partial N}\right|_V = \frac{1}{V}. \tag{6}$$

Equation (5) is calculated as

$$\begin{aligned} \frac{\partial E}{\partial V} &= f(\rho) + Vf'(\rho)\left(-\frac{N}{V^2}\right) && (7)\\ &= f(\rho) - \rho f'(\rho). && (8) \end{aligned}$$

Then,

$$\begin{aligned} \frac{\partial^2 E}{\partial V^2} &= f'(\rho)\frac{\partial \rho}{\partial V} - \frac{\partial \rho}{\partial V}f'(\rho) - \rho f''(\rho)\frac{\partial \rho}{\partial V} && (9)\\ &= -\rho f''(\rho)\frac{\partial \rho}{\partial V} && (10)\\ &= -\left(\frac{N}{V}\right)f''(\rho)\left(-\frac{N}{V^2}\right) && (11)\\ &= \frac{1}{V}f''(\rho)\left(\frac{N}{V}\right)^2. && (12) \end{aligned}$$

Next, let us consider the chemical potential $\mu \equiv \partial E/\partial N$, $(T \sim 0)$. From the assumption in Eq. (4),

$$\begin{aligned} \mu &= \frac{\partial E}{\partial N} && (13)\\ &= Vf'(\rho)\frac{\partial \rho}{\partial N} && (14)\\ &= f'(\rho), && (15) \end{aligned}$$

where we have utilized Eq. (6) in the last line. Then,

$$\frac{\partial \mu}{\partial N} = f''(\rho)\frac{\partial \rho}{\partial N} \tag{16}$$

$$= \frac{1}{V} f''(\rho). \tag{17}$$

From Eqs. (12) and (17),

$$\frac{\partial^2 E}{\partial V^2} = \frac{1}{V} f''(\rho)\left(\frac{N}{V}\right)^2 \tag{18}$$

$$= \frac{\partial \mu}{\partial N}\left(\frac{N}{V}\right)^2 \tag{19}$$

Substituting this into the expression for κ^{-1} in Eq. (3),

$$\kappa^{-1} = V\frac{\partial^2 E}{\partial V^2} \tag{20}$$

$$= V\frac{\partial \mu}{\partial N}\left(\frac{N}{V}\right)^2 \tag{21}$$

$$= \frac{N^2}{V}\frac{\partial \mu}{\partial N}. \tag{22}$$

Thus, we obtain the final result:

$$\kappa^{-1} = N\rho\frac{\partial \mu}{\partial N}. \tag{23}$$

1.2 *First Step:*

Let us consider the potential-energy term of the Hamiltonian:

$$H_{\rm PE} = \sum_{k,k'} \tilde{V}_{kk'} c^\dagger_{k,\uparrow} c^\dagger_{-k,\downarrow} c_{-k',\downarrow} c_{k',\uparrow}. \tag{24}$$

We define the operator b_k as

$$b_k = c_{-k,\downarrow} c_{k,\uparrow}, \tag{25}$$

and then its thermal average is

$$\langle b_k \rangle = \langle c_{-k,\downarrow} c_{k,\uparrow} \rangle, \tag{26}$$

where $\langle \cdots \rangle = {\rm Tr}[e^{-\beta H} \cdots]/{\rm Tr}[e^{-\beta H}]$ and $\beta = 1/k_{\rm B}T$. In the superconducting state, $\langle b_k \rangle \neq 0$. We define the deviation δb_k of b_k from its thermal average $\langle b_k \rangle$ as

$$\delta b_k = b_k - \langle b_k \rangle, \tag{27}$$

namely

$$b_k = \langle b_k \rangle + \delta b_k. \tag{28}$$

Substituting this into $H_{\rm PE}$,

$$\begin{aligned}
H_{\rm PE} &= \sum_{k,k'} \tilde{V}_{kk'} c^\dagger_{k,\uparrow} c^\dagger_{-k,\downarrow} c_{-k',\downarrow} c_{k',\uparrow} && (29)\\
&= \sum_{k,k'} \tilde{V}_{kk'} b^\dagger_k b_{k'} && (30)\\
&= \sum_{k,k'} \tilde{V}_{kk'} \Big(\langle b^\dagger_k \rangle + \delta b^\dagger_k \Big) \Big(\langle b_{k'} \rangle + \delta b_{k'} \Big) && (31)\\
&= \sum_{k,k'} \tilde{V}_{kk'} \Big(\langle b^\dagger_k \rangle \langle b_{k'} \rangle + \langle b^\dagger_k \rangle \delta b_{k'} + \delta b^\dagger_k \langle b_{k'} \rangle + \delta b^\dagger_k \delta b_{k'} \Big). && (32)
\end{aligned}$$

Here, we assume the deviation δb_k is small, and we neglect the deviation-square term $\delta b^\dagger_k \delta b_{k'}$ in $H_{\rm PE}$ (the mean-field approximation):

$$H_{\rm PE} \approx \sum_{k,k'} \tilde{V}_{kk'} \Big(\langle b^\dagger_k \rangle \langle b_{k'} \rangle + \langle b^\dagger_k \rangle \delta b_{k'} + \delta b^\dagger_k \langle b_{k'} \rangle \Big). \tag{33}$$

Then, we proceed on a calculation,

$$\begin{aligned}
H_{\rm PE} &\approx \sum_{k,k'} \tilde{V}_{kk'} \Big(\langle b^\dagger_k \rangle \langle b_{k'} \rangle + \langle b^\dagger_k \rangle \delta b_{k'} + \delta b^\dagger_k \langle b_{k'} \rangle \Big) && (34)\\
&= \sum_{k,k'} \tilde{V}_{kk'} \Bigg[\langle b^\dagger_k \rangle \langle b_{k'} \rangle + \langle b^\dagger_k \rangle \Big(b_{k'} - \langle b_{k'} \rangle \Big) + \Big(b^\dagger_k - \langle b^\dagger_k \rangle \Big) \langle b_{k'} \rangle \Bigg] && (35)\\
&= \sum_{k,k'} \tilde{V}_{kk'} \Bigg[\langle b^\dagger_k \rangle \langle b_{k'} \rangle - 2 \langle b^\dagger_k \rangle \langle b_{k'} \rangle + \langle b^\dagger_k \rangle b_{k'} + b^\dagger_k \langle b_{k'} \rangle \Bigg] && (36)\\
&= \sum_{k,k'} \tilde{V}_{kk'} \Bigg[\langle b^\dagger_k \rangle b_{k'} + b^\dagger_k \langle b_{k'} \rangle - \langle b^\dagger_k \rangle \langle b_{k'} \rangle \Bigg] && (37)\\
&= -\sum_{k} \Bigg[\Delta^*_k b_k + b^\dagger_k \Delta_k - \Delta^*_k \langle b_k \rangle \Bigg], && (38)
\end{aligned}$$

where we have defined

$$\Delta_k = \sum_{k'}(-\tilde{V}_{kk'})\langle b_{k'}\rangle, \qquad \text{and} \qquad \Delta_k^* = \sum_{k'}(-\tilde{V}_{k'k})\langle b_{k'}^\dagger\rangle = \sum_{k'}(-\tilde{V}_{kk'}^*)\langle b_{k'}^\dagger\rangle. \tag{39}$$

Here $\tilde{V}_{k'k} = \tilde{V}_{kk'}^*$ because $H_{\rm PE}^\dagger = H_{\rm PE}$. The superconducting state is characterized by $\Delta_k \neq 0$, ($\Delta_k = 0$ in the normal state).

Second Step:

To diagonalize the resulting Hamiltonian, let us consider the so-called Bogoliubov-Valatin transformation:

$$c_{k,\uparrow} = u_k\gamma_{k0} + v_k^*\gamma_{k1}^\dagger, \tag{40}$$

$$c_{-k,\downarrow}^\dagger = -v_k\gamma_{k0} + u_k^*\gamma_{k1}^\dagger, \tag{41}$$

where $|u_k|^2+|v_k|^2 = 1$, and the new operators γ_{k0} and γ_{k1} follow the relations for the Fermion operators: $\{\gamma_{ki}, \gamma_{k'j}^\dagger\} = \delta_{k,k'}\delta_{i,j}$ and $\{\gamma_{ki}, \gamma_{k'j}\} = \{\gamma_{ki}^\dagger, \gamma_{k'j}^\dagger\} = 0$. We calculate,

$$\begin{aligned}
b_k &= c_{-k,\downarrow}c_{k,\uparrow} && (42)\\
&= (-v_k^*\gamma_{k0}^\dagger + u_k\gamma_{k1})(u_k\gamma_{k0} + v_k^*\gamma_{k1}^\dagger) && (43)\\
&= -v_k^*u_k\gamma_{k0}^\dagger\gamma_{k0} - v_k^*v_k^*\gamma_{k0}^\dagger\gamma_{k1}^\dagger + u_ku_k\gamma_{k1}\gamma_{k0} + u_kv_k^*\gamma_{k1}\gamma_{k1}^\dagger && (44)\\
&= -v_k^*u_k\gamma_{k0}^\dagger\gamma_{k0} - v_k^*v_k^*\gamma_{k0}^\dagger\gamma_{k1}^\dagger + u_ku_k\gamma_{k1}\gamma_{k0} + u_kv_k^*(1 - \gamma_{k1}^\dagger\gamma_{k1}), && (45)
\end{aligned}$$

and

$$\begin{aligned}
b_k^\dagger &= (b_k)^\dagger && (46)\\
&= -v_ku_k^*\gamma_{k0}^\dagger\gamma_{k0} - v_kv_k\gamma_{k1}\gamma_{k0} + u_k^*u_k^*\gamma_{k0}^\dagger\gamma_{k1}^\dagger + u_k^*v_k(1 - \gamma_{k1}^\dagger\gamma_{k1}). && (47)
\end{aligned}$$

Substituting these into $H_{\rm PE}$ in Eq. (38),

$$\begin{aligned}
H_{\rm PE} &= -\sum_k\Big[\Delta_k^*b_k + b_k^\dagger\Delta_k - \Delta_k^*\langle b_k\rangle\Big] && (48)\\
&= -\sum_k \\
&\quad\times\Big[\Delta_k^*\Big\{-v_k^*u_k\gamma_{k0}^\dagger\gamma_{k0} - v_k^*v_k^*\gamma_{k0}^\dagger\gamma_{k1}^\dagger + u_ku_k\gamma_{k1}\gamma_{k0} + u_kv_k^*(1 - \gamma_{k1}^\dagger\gamma_{k1})\Big\}\\
&\qquad + \Delta_k\Big\{-v_ku_k^*\gamma_{k0}^\dagger\gamma_{k0} - v_kv_k\gamma_{k1}\gamma_{k0} + u_k^*u_k^*\gamma_{k0}^\dagger\gamma_{k1}^\dagger + u_k^*v_k(1 - \gamma_{k1}^\dagger\gamma_{k1})\Big\}\\
&\qquad - \Delta_k^*\langle b_k\rangle\Big] && (49)\\
&= -\sum_k \\
&\quad\times\Big[\Big\{-\Delta_k^*v_k^*u_k - \Delta_kv_ku_k^*\Big\}\gamma_{k0}^\dagger\gamma_{k0} + \Big\{-\Delta_k^*v_k^*v_k^* + \Delta_ku_k^*u_k^*\Big\}\gamma_{k0}^\dagger\gamma_{k1}^\dagger\\
&\qquad + \Big\{\Delta_k^*u_ku_k - \Delta_kv_kv_k\Big\}\gamma_{k1}\gamma_{k0} + \Big\{\Delta_k^*u_kv_k^* + \Delta_ku_k^*v_k\Big\}(1 - \gamma_{k1}^\dagger\gamma_{k1})\\
&\qquad - \Delta_k^*\langle b_k\rangle\Big]. && (50)
\end{aligned}$$

As for the kinetic-energy term $H_{\rm KE}$ of the Hamiltonian,

$$\begin{aligned}
H_{\rm KE} &= \sum_{k,s} \xi_k c^\dagger_{k,s} c_{k,s} && (51)\\
&= \sum_{k} \xi_k (c^\dagger_{k,\uparrow} c_{k,\uparrow} + c^\dagger_{k,\downarrow} c_{k,\downarrow}) && (52)\\
&= \sum_{k} \xi_k (c^\dagger_{k,\uparrow} c_{k,\uparrow} + c^\dagger_{-k,\downarrow} c_{-k,\downarrow}), \qquad (\xi_{-k} = \xi_k) && (53)\\
&= \sum_{k} \xi_k \Big[(u_k^* \gamma^\dagger_{k0} + v_k \gamma_{k1})(u_k \gamma_{k0} + v_k^* \gamma^\dagger_{k1}) + (-v_k \gamma_{k0} + u_k^* \gamma^\dagger_{k1})(-v_k^* \gamma^\dagger_{k0} + u_k \gamma_{k1})\Big] && (54)\\
&= \sum_{k} \xi_k \Big[(u_k^* u_k \gamma^\dagger_{k0}\gamma_{k0} + u_k^* v_k^* \gamma^\dagger_{k0}\gamma^\dagger_{k1} + v_k u_k \gamma_{k1}\gamma_{k0} + v_k v_k^* \gamma_{k1}\gamma^\dagger_{k1})\\
&\qquad + (v_k v_k^* \gamma_{k0}\gamma^\dagger_{k0} - v_k u_k \gamma_{k0}\gamma_{k1} - u_k^* v_k^* \gamma^\dagger_{k1}\gamma^\dagger_{k0} + u_k^* u_k \gamma^\dagger_{k1}\gamma_{k1})\Big] && (55)\\
&= \sum_{k} \xi_k \Big[|u_k|^2 \gamma^\dagger_{k0}\gamma_{k0} + |v_k|^2 (1 - \gamma^\dagger_{k0}\gamma_{k0}) + 2u_k^* v_k^* \gamma^\dagger_{k0}\gamma^\dagger_{k1}\\
&\qquad + 2 v_k u_k \gamma_{k1}\gamma_{k0} + |v_k|^2 (1 - \gamma^\dagger_{k1}\gamma_{k1}) + |u_k|^2 \gamma^\dagger_{k1}\gamma_{k1})\Big] && (56)\\
& && (57)
\end{aligned}$$

Hence, the Hamiltonian $H = H_{\rm KE} + H_{\rm PE}$ is

$$\begin{aligned}
H &= H_{\rm KE} + H_{\rm PE} && (58)\\
&= \sum_{k} \xi_k \Big[|u_k|^2 \gamma^\dagger_{k0}\gamma_{k0} + |v_k|^2 (1 - \gamma^\dagger_{k0}\gamma_{k0}) + 2u_k^* v_k^* \gamma^\dagger_{k0}\gamma^\dagger_{k1}\\
&\qquad + 2 v_k u_k \gamma_{k1}\gamma_{k0} + |v_k|^2 (1 - \gamma^\dagger_{k1}\gamma_{k1}) + |u_k|^2 \gamma^\dagger_{k1}\gamma_{k1})\Big]\\
&\quad - \sum_{k} \Big[(-\Delta_k^* v_k^* u_k - \Delta_k v_k u_k^*) \gamma^\dagger_{k0}\gamma_{k0} + (-\Delta_k^* v_k^* v_k^* + \Delta_k u_k^* u_k^*) \gamma^\dagger_{k0}\gamma^\dagger_{k1}\\
&\qquad + (\Delta_k^* u_k u_k - \Delta_k v_k v_k) \gamma_{k1}\gamma_{k0} + (\Delta_k^* u_k v_k^* + \Delta_k u_k^* v_k)(1 - \gamma^\dagger_{k1}\gamma_{k1})\\
&\qquad - \Delta_k^* \langle b_k \rangle\Big] && (59)\\
&= \sum_{k} \Big[\xi_k |u_k|^2 \gamma^\dagger_{k0}\gamma_{k0} + \xi_k |v_k|^2 (1 - \gamma^\dagger_{k0}\gamma_{k0}) - (-\Delta_k^* v_k^* u_k - \Delta_k v_k u_k^*) \gamma^\dagger_{k0}\gamma_{k0}\\
&\qquad + 2\xi_k u_k^* v_k^* \gamma^\dagger_{k0}\gamma^\dagger_{k1} - (-\Delta_k^* v_k^* v_k^* + \Delta_k u_k^* u_k^*) \gamma^\dagger_{k0}\gamma^\dagger_{k1}\\
&\qquad + 2\xi_k v_k u_k \gamma_{k1}\gamma_{k0} - (\Delta_k^* u_k u_k - \Delta_k v_k v_k) \gamma_{k1}\gamma_{k0}\\
&\qquad + \xi_k |v_k|^2 (1 - \gamma^\dagger_{k1}\gamma_{k1}) + \xi_k |u_k|^2 \gamma^\dagger_{k1}\gamma_{k1} - (\Delta_k^* u_k v_k^* + \Delta_k u_k^* v_k)(1 - \gamma^\dagger_{k1}\gamma_{k1})\\
&\qquad + \Delta_k^* \langle b_k \rangle\Big] && (60)\\
&= \sum_{k} \Big[\xi_k (|u_k|^2 - |v_k|^2) \gamma^\dagger_{k0}\gamma_{k0} + (\Delta_k^* v_k^* u_k + \Delta_k v_k u_k^*) \gamma^\dagger_{k0}\gamma_{k0}\\
&\qquad + 2\xi_k u_k^* v_k^* \gamma^\dagger_{k0}\gamma^\dagger_{k1} + (\Delta_k^* v_k^* v_k^* - \Delta_k u_k^* u_k^*) \gamma^\dagger_{k0}\gamma^\dagger_{k1}\\
&\qquad + 2\xi_k v_k u_k \gamma_{k1}\gamma_{k0} + (-\Delta_k^* u_k u_k + \Delta_k v_k v_k) \gamma_{k1}\gamma_{k0}\\
&\qquad + \xi_k (|u_k|^2 - |v_k|^2) \gamma^\dagger_{k1}\gamma_{k1} + (\Delta_k^* u_k v_k^* + \Delta_k u_k^* v_k) \gamma^\dagger_{k1}\gamma_{k1}\\
&\qquad + \Delta_k^* \langle b_k \rangle + 2\xi_k |v_k|^2 - (\Delta_k^* u_k v_k^* + \Delta_k u_k^* v_k)\Big] && (61)
\end{aligned}$$

$$
\begin{aligned}
= \sum_k \Big[& \Big\{ \xi_k(|u_k|^2 - |v_k|^2) + (\Delta_k^* v_k^* u_k + \Delta_k v_k u_k^*) \Big\} \gamma_{k0}^\dagger \gamma_{k0} \\
& + \Big\{ 2\xi_k u_k^* v_k^* + (\Delta_k^* v_k^* v_k^* - \Delta_k u_k^* u_k^*) \Big\} \gamma_{k0}^\dagger \gamma_{k1}^\dagger \\
& + \Big\{ 2\xi_k v_k u_k + (-\Delta_k^* u_k u_k + \Delta_k v_k v_k) \Big\} \gamma_{k1} \gamma_{k0} \\
& + \Big\{ \xi_k(|u_k|^2 - |v_k|^2) + (\Delta_k^* u_k v_k^* + \Delta_k u_k^* v_k) \Big\} \gamma_{k1}^\dagger \gamma_{k1} \\
& + \Delta_k^* \langle b_k \rangle + 2\xi_k |v_k|^2 - (\Delta_k^* u_k v_k^* + \Delta_k u_k^* v_k) \Big]. \qquad (62)
\end{aligned}
$$

This Hamiltonian is diagonalized as

$$
\begin{aligned}
H = \sum_k \Big[& \Big\{ \xi_k(|u_k|^2 - |v_k|^2) + (\Delta_k^* v_k^* u_k + \Delta_k v_k u_k^*) \Big\} \gamma_{k0}^\dagger \gamma_{k0} \\
& + \Big\{ \xi_k(|u_k|^2 - |v_k|^2) + (\Delta_k^* u_k v_k^* + \Delta_k u_k^* v_k) \Big\} \gamma_{k1}^\dagger \gamma_{k1} \\
& + \Delta_k^* \langle b_k \rangle + 2\xi_k |v_k|^2 - (\Delta_k^* u_k v_k^* + \Delta_k u_k^* v_k) \Big], \qquad (63)
\end{aligned}
$$

if the coefficients of the operators $\gamma_{k0}^\dagger \gamma_{k1}^\dagger$ and $\gamma_{k1}\gamma_{k0}$ are zero in Eq. (62):

$$
2\xi_k u_k^* v_k^* + (\Delta_k^* v_k^* v_k^* - \Delta_k u_k^* u_k^*) = 0 \qquad (64)
$$

$$
2\xi_k v_k u_k + (-\Delta_k^* u_k u_k + \Delta_k v_k v_k) = 0. \qquad (65)
$$

The above two equations (64) and (65) are equivalent:

$$
2\xi_k v_k u_k + \Delta_k v_k v_k - \Delta_k^* u_k u_k = 0. \qquad (66)
$$

Multiplying this by $v_k^* u_k^*$ gives

$$
2\xi_k |v_k|^2 |u_k|^2 + \Delta_k |v_k|^2 u_k^* v_k - \Delta_k^* |u_k|^2 u_k v_k^* = 0. \qquad (67)
$$

Its complex conjugate is

$$
2\xi_k |v_k|^2 |u_k|^2 + \Delta_k^* |v_k|^2 u_k v_k^* - \Delta_k |u_k|^2 u_k^* v_k = 0. \qquad (68)
$$

From (67) + (68),

$$
4\xi_k |v_k|^2 |u_k|^2 + \Delta_k(|v_k|^2 - |u_k|^2) u_k^* v_k + \Delta_k^*(|v_k|^2 - |u_k|^2) u_k v_k^* = 0. \qquad (69)
$$

Defining the phases ϕ and ψ as

$$
\Delta_k = |\Delta_k| \exp(i\phi) \qquad \text{and} \qquad u_k v_k^* = |u_k v_k^*| \exp(i\psi) = |u_k||v_k| \exp(i\psi), \qquad (70)
$$

we have

$$
4\xi_k |v_k|^2 |u_k|^2 + |\Delta_k||u_k||v_k|(|v_k|^2 - |u_k|^2) e^{i(\phi-\psi)} + |\Delta_k||u_k||v_k|(|v_k|^2 - |u_k|^2) e^{-i(\phi-\psi)} = 0. \qquad (71)
$$

$\rightarrow$

$$
4\xi_k |v_k||u_k| + |\Delta_k|(|v_k|^2 - |u_k|^2) e^{i(\phi-\psi)} + |\Delta_k|(|v_k|^2 - |u_k|^2) e^{-i(\phi-\psi)} = 0. \qquad (72)
$$

$\rightarrow$

$$2\xi_k|v_k||u_k| + |\Delta_k|\cos(\phi - \psi)(|v_k|^2 - |u_k|^2) = 0. \tag{73}$$

Considering the assumption $|u_k|^2 + |v_k|^2 = 1$, we set

$$|u_k| = \cos t \qquad \text{and} \qquad |v_k| = \sin t. \tag{74}$$

We also define

$$\tilde{\Delta}_k = |\Delta_k|\cos(\phi - \psi). \tag{75}$$

(We will later find $\phi = \psi$.) From Eq. (73),

$$\begin{aligned} 0 &= 2\xi_k \cos t \sin t + \tilde{\Delta}_k(\sin^2 t - \cos^2 t) && (76) \\ &= \xi_k \sin(2t) - \tilde{\Delta}_k \cos(2t). && (77) \end{aligned}$$

$\rightarrow$

$$\tan(2t) = \frac{\tilde{\Delta}_k}{\xi_k}. \tag{78}$$

Therefore,

$$\cos(2t) = \frac{\xi_k}{E_k} \qquad \text{and} \qquad \sin(2t) = \frac{\tilde{\Delta}_k}{E_k}, \tag{79}$$

where we have defined

$$E_k = \sqrt{\xi_k^2 + \tilde{\Delta}_k^2}. \tag{80}$$

Thus,

$$\begin{aligned} |u_k|^2 &= \cos^2 t && (81) \\ &= \frac{1}{2}\Big(1 + \cos(2t)\Big) && (82) \\ &= \frac{1}{2}\left(1 + \frac{\xi_k}{E_k}\right), && (83) \end{aligned}$$

and

$$\begin{aligned} |v_k|^2 &= \sin^2 t && (84) \\ &= \frac{1}{2}\Big(1 - \cos(2t)\Big) && (85) \\ &= \frac{1}{2}\left(1 - \frac{\xi_k}{E_k}\right). && (86) \end{aligned}$$

The expressions for $|u_k|^2$ and $|v_k|^2$ in Eqs. (83) and (86) have been obtained from the condition [Eq. (66)] for diagonalizing the Hamiltonian.

From the diagonalized Hamiltonian in Eq. (63),

$$\begin{aligned} H &= \sum_k \Big[\Big\{ \xi_k(|u_k|^2 - |v_k|^2) + (\Delta_k^* v_k^* u_k + \Delta_k v_k u_k^*) \Big\} \gamma_{k0}^\dagger \gamma_{k0} \\ &\qquad + \Big\{ \xi_k(|u_k|^2 - |v_k|^2) + (\Delta_k^* u_k v_k^* + \Delta_k u_k^* v_k) \Big\} \gamma_{k1}^\dagger \gamma_{k1} \\ &\qquad + \Delta_k^* \langle b_k \rangle + 2\xi_k |v_k|^2 - (\Delta_k^* u_k v_k^* + \Delta_k u_k^* v_k) \Big] \qquad (87) \\ &= \sum_k \Big[\Big\{ \frac{\xi_k^2}{E_k} + \frac{\tilde{\Delta}_k^2}{E_k} \Big\} \gamma_{k0}^\dagger \gamma_{k0} \\ &\qquad + \Big\{ \frac{\xi_k^2}{E_k} + \frac{\tilde{\Delta}_k^2}{E_k} \Big\} \gamma_{k1}^\dagger \gamma_{k1} \\ &\qquad + \Delta_k^* \langle b_k \rangle + 2\xi_k |v_k|^2 - (\Delta_k^* u_k v_k^* + \Delta_k u_k^* v_k) \Big] \qquad (88) \\ &= \sum_k \Big[E_k \gamma_{k0}^\dagger \gamma_{k0} + E_k \gamma_{k1}^\dagger \gamma_{k1} \Big] + E_0. \qquad (89) \end{aligned}$$

Here, the excitation energy is E_k, and the ground state energy is

$$E_0 \equiv \sum_k \Big[\Delta_k^* \langle b_k \rangle + 2\xi_k |v_k|^2 - (\Delta_k^* u_k v_k^* + \Delta_k u_k^* v_k) \Big]. \qquad (90)$$

In Eq. (88), we referred to Eqs. (70), (83) and (86).

Third Step:

Let us consider the thermal average of $b_k = c_{-k,\downarrow} c_{k,\uparrow}$. From Eq. (45),

$$\begin{aligned} \langle b_k \rangle &= \langle c_{-k,\downarrow} c_{k,\uparrow} \rangle \qquad (91) \\ &= -v_k^* u_k \langle \gamma_{k0}^\dagger \gamma_{k0} \rangle - v_k^* v_k^* \langle \gamma_{k0}^\dagger \gamma_{k1}^\dagger \rangle + u_k u_k \langle \gamma_{k1} \gamma_{k0} \rangle + u_k v_k^* (1 - \langle \gamma_{k1}^\dagger \gamma_{k1} \rangle) \qquad (92) \\ &= u_k v_k^* \Big\{ 1 - 2f(E_k) \Big\} \qquad (93) \\ &= u_k v_k^* \tanh\Big(\frac{\beta E_k}{2} \Big) \qquad (94) \\ &= |u_k||v_k| e^{i\psi} \tanh\Big(\frac{\beta E_k}{2} \Big) \qquad (95) \\ &= \frac{|\tilde{\Delta}_k|}{2E_k} e^{i\psi} \tanh\Big(\frac{\beta E_k}{2} \Big). \qquad (96) \end{aligned}$$

Here, $\langle \gamma_{k0}^\dagger \gamma_{k0} \rangle = \langle \gamma_{k1}^\dagger \gamma_{k1} \rangle = f(E_k)$ and $\langle \gamma_{k0}^\dagger \gamma_{k1}^\dagger \rangle = \langle \gamma_{k1} \gamma_{k0} \rangle = 0$, because $\gamma_{k0}^\dagger$ and $\gamma_{k1}^\dagger$ are the independent Fermionic operators for the excited states with the energy E_k according to Eq. (89). [$f(E_k)$ is the Fermi distribution function.]

Substituting Eq. (96) into the expression for Δ_k of Eq. (39),

$$\begin{aligned} \Delta_k &= \sum_{k'} (-\tilde{V}_{kk'}) \langle b_{k'} \rangle \qquad (97) \\ &= \sum_{k'} (-\tilde{V}_{kk'}) \frac{|\tilde{\Delta}_{k'}|}{2E_{k'}} e^{i\psi} \tanh\Big(\frac{\beta E_{k'}}{2} \Big). \qquad (98) \end{aligned}$$

Now, referring to Eq. (70),

$$|\Delta_k|e^{\phi} = \sum_{k'}(-\tilde{V}_{kk'})\langle b_{k'}\rangle \tag{99}$$

$$= \sum_{k'}(-\tilde{V}_{kk'})\frac{|\tilde{\Delta}_{k'}|}{2E_{k'}}e^{i\psi}\tanh\Big(\frac{\beta E_{k'}}{2}\Big). \tag{100}$$

$\rightarrow$

$$|\Delta_k| = \sum_{k'}(-\tilde{V}_{kk'})e^{-i(\phi-\psi)}\frac{|\tilde{\Delta}_{k'}|}{2E_{k'}}\tanh\Big(\frac{\beta E_{k'}}{2}\Big). \tag{101}$$

Therefore, $\phi = \psi$ when $\tilde{V}_{kk'}$ is real. ($\tilde{V}_{kk'} < 0$.) Hence we set $\phi = \psi$. Then, $\tilde{\Delta}_{k'} = |\Delta_{k'}|$ from Eq. (75), $E_k = \sqrt{\xi_k^2 + |\Delta_k|^2}$ from Eq. (80), and we get the gap equation from Eq. (98) finally,

$$\Delta_k = \sum_{k'}(-\tilde{V}_{kk'})\frac{\Delta_{k'}}{2E_{k'}}\tanh\Big(\frac{\beta E_{k'}}{2}\Big). \tag{102}$$

Fourth Step:
At $T \rightarrow T_c$, we linearize the above gap equation as

$$\Delta_k = \sum_{k'}(-\tilde{V}_{kk'})\frac{\Delta_{k'}}{2|\xi_{k'}|}\tanh\Big(\frac{\beta|\xi_{k'}|}{2}\Big). \tag{103}$$

Here, we have expanded $E_k = \sqrt{\xi_k^2 + |\Delta_k|^2}$ with respect to $|\Delta_k|/\xi_k$, and have kept only its zeroth order.

Now, let us define

$$\Delta_k \equiv \Delta(\xi_k)g_k, \tag{104}$$

where

$$\int \frac{d\Omega_k}{4\pi}|g_k|^2 = 1. \tag{105}$$

Substituting this into the linearized gap equation,

$$\Delta(\xi_k)g_k = \sum_{k'}(-\tilde{V}_{kk'})g_{k'}\frac{\Delta(\xi_{k'})}{2|\xi_{k'}|}\tanh\Big(\frac{\beta|\xi_{k'}|}{2}\Big). \tag{106}$$

$\rightarrow$

$$\Delta(\xi_k)\int\frac{d\Omega_k}{4\pi}|g_k|^2 = -\sum_{k'}\int\frac{d\Omega_k}{4\pi}g_k^*\tilde{V}_{kk'}g_{k'}\frac{\Delta(\xi_{k'})}{2|\xi_{k'}|}\tanh\Big(\frac{\beta|\xi_{k'}|}{2}\Big). \tag{107}$$

$\rightarrow$

$$\Delta(\xi_k) = -\Big(\int\frac{d^3k'}{(2\pi)^3}\Big)\int\frac{d\Omega_k}{4\pi}g_k^*\tilde{V}_{kk'}g_{k'}\frac{\Delta(\xi_{k'})}{2|\xi_{k'}|}\tanh\Big(\frac{\beta|\xi_{k'}|}{2}\Big) \tag{108}$$

$$= -\Big(N_0\int_{-W}^{W}d\xi_{k'}\int\frac{d\Omega_{k'}}{4\pi}\Big)\int\frac{d\Omega_k}{4\pi}g_k^*\tilde{V}_{kk'}g_{k'}\frac{\Delta(\xi_{k'})}{2|\xi_{k'}|}\tanh\Big(\frac{\beta|\xi_{k'}|}{2}\Big) \tag{109}$$

$$= -N_0\int_{-W}^{W}d\xi_{k'}\tilde{V}(\xi_k,\xi_{k'})\frac{\Delta(\xi_{k'})}{2|\xi_{k'}|}\tanh\Big(\frac{\beta|\xi_{k'}|}{2}\Big) \tag{110}$$

$$= -N_0\int_{\xi_{k'}>0}d\xi_{k'}\tilde{V}(\xi_k,\xi_{k'})\frac{\Delta(\xi_{k'})}{\xi_{k'}}\tanh\Big(\frac{\beta\xi_{k'}}{2}\Big), \tag{111}$$

where N_0 is the density of states, we have defined

$$\tilde{V}(\xi_k, \xi_{k'}) \equiv \int \frac{d\Omega_{k'}}{4\pi} \int \frac{d\Omega_k}{4\pi} g_k^* \tilde{V}_{kk'} g_{k'}, \tag{112}$$

and we have assumed $\Delta(\xi_k)$ is an even function of ξ_k in Eq. (111).