Unkonventionelle Supraleitung WS 05/06
Lösungen zur Serie 2

2.1 Assuming that liquid ^{3}He — of atomic mass m_3 — may be described as a Fermi gas (FG), with a molar volume of 37 cm^3, we estimate the following:

ia) The density ρ of ^{3}He, Fermi wave number k_F, Fermi energy E_F, density of states $N(E_\mathrm{F}) = 2D(E_\mathrm{F})$, and Fermi temperature T_F

$$\rho = \frac{N}{V} = \frac{N_\mathrm{A}}{37\mathrm{cm}^3} = \frac{6.022\times 10^{23}}{37\mathrm{cm}^3} = 0.1628\times 10^{23}\mathrm{cm}^{-3} \quad (1)$$

$$k_\mathrm{F} = (3\pi^2\rho)^{1/3} = 7.838\times 10^7\mathrm{cm}^{-1} \quad (2)$$

$$E_\mathrm{F} = \frac{\hbar^2 k_\mathrm{F}^2}{2m_3} = 6.827\times 10^{-16}\mathrm{erg} \quad (3)$$

$$N(E_\mathrm{F}) = \frac{3\rho}{2E_\mathrm{F}} = 3.577\times 10^{37}\mathrm{states}\cdot\mathrm{cm}^{-3}\cdot\mathrm{erg}^{-1} \quad (4)$$

$$\text{or}\qquad N(E_\mathrm{F}) = \frac{3N_\mathrm{A}}{2E_\mathrm{F}} = 1.323\times 10^{39}\mathrm{states}\cdot\mathrm{mol}^{-1}\cdot\mathrm{erg}^{-1} \quad (5)$$

$$T_\mathrm{F} = \frac{E_\mathrm{F}}{k_\mathrm{B}} \approx 4.9\mathrm{K}. \quad (6)$$

ib) Magnetic susceptibility per unit volume in cgs units

$$\chi_\mathrm{FG} = \beta^2 N(E_\mathrm{F}) \approx 3.6\times 10^{37}\beta^2. \quad (7)$$

Here, β is the ^{3}He nuclear magnetic moment in cgs units (β= 2.13 nuclear magnetons = 2.13×10^{-24} erg/Gauss).

ic) Specific heat C_FG (with $R = N_\mathrm{A} k_\mathrm{B}$, the gas constant)

$$C_\mathrm{FG}(\text{per mol}) = \frac{\pi^2}{3} k_\mathrm{B}^2 N(E_\mathrm{F}) T. \quad (8)$$

$$\rightarrow \qquad \frac{C_\mathrm{FG}}{RT} = \frac{\pi^2}{3} k_\mathrm{B}^2 \Big(\frac{3N_\mathrm{A}}{2E_\mathrm{F}}\Big)\frac{1}{N_\mathrm{A}k_\mathrm{B}} = \frac{\pi^2}{2}\frac{k_\mathrm{B}}{E_\mathrm{F}} \approx 1\mathrm{K}^{-1}. \quad (9)$$

ic) Sound velocity S_FG

$$S_\mathrm{FG}^2 = \frac{1}{\kappa m_3\rho} = \frac{1}{m_3\rho} N\rho\frac{\partial\mu}{\partial N} = \frac{N}{m_3}\frac{\partial E_\mathrm{F}}{\partial N} = \frac{N}{m_3}\frac{\hbar^2}{2m_3}\frac{\partial}{\partial N}\Big(3\pi^2\frac{N}{V}\Big)^{\frac{2}{3}}$$

$$= \frac{1}{m_3}\frac{\hbar^2}{2m_3}\frac{2}{3}\Big(3\pi^2\frac{N}{V}\Big)^{\frac{2}{3}} = \frac{1}{m_3}\frac{2}{3}E_\mathrm{F} = \frac{2}{3}\frac{p_\mathrm{F}^2}{2m_3^2} \quad (10)$$

$$\rightarrow \qquad S_\mathrm{FG} = 9533\mathrm{cm}\cdot\mathrm{s}^{-1} \approx 95\mathrm{m}\cdot\mathrm{s}^{-1} \quad (11)$$

A comparison of the estimates with the observed low-temperature (between 2 and 100 mK) values at atmospheric pressure, is given in the following table.

	Fermi gas (FG)	Experimental (Exp)	Ratio (Exp/FG)
$T_{\rm F}$ (K)	4.9	1.77	0.36
χ (cgs)	$3.6\times10^{37}\beta^2$	$3.3\times10^{38}\beta^2$	9.2
C/RT ($\rm K^{-1}$)	1	2.78	2.8
S ($\rm ms^{-1}$)	95	183	1.9

Here, the experimental $T_{\rm F}$ is obtained from the relation $C/RT = \pi^2/2T_{\rm F}$.

Because of the significant discrepancies between the experimental results and the estimates, one concludes that the Fermi gas model is not satisfactory for the liquid ^{3}He. The interactions between the ^{3}He atoms have to be taken into account (the Fermi-liquid effect).

ii) How well localized in real space are the ^{3}He atoms? We use two rough estimates.
a) From $\Delta x \cdot \Delta k \sim 1$ with $\Delta k \sim k_{\rm F}$ it follows $\Delta x \sim 1$Å.
b) Δx is identified with the de Broglie wavelength for a particle with a mass of m_3, the ^{3}He atom mass and momentum $p = (2m_3 k_{\rm B} T_{\rm F})^{1/2}$. One finds $\Delta x \sim \hbar/p \sim 1$Å. We conclude that the ^{3}He atoms are delocalized on the scale of inter-atomic distances. Since further localization in real space implies an extra cost of kinetic energy, the total energy of an ensemble of ^{3}He atoms reaches a minimum at considerably greater atomic volume than expected from the minimum of the Lennard-Jones potential. This results for ^{3}He in a very shallow potential, with a large compressibility and a kinetic energy dominated by quantum effects. Therefore, the liquid does not solidify at ambient pressure.

2.2 Considering liquid ^{3}He to be a Fermi liquid. (see the above table.)
i) The effective mass is $m_3^*/m_3 = C_{\rm exp}/C_{\rm FG} = 2.8$.
ii) The Fermi-liquid parameters F_1 and Z_0 are as follows.

$$\Big(1+\frac{F_1}{3}\Big) = \frac{m_3^*}{m_3} = 2.8 \qquad \rightarrow \qquad F_1 = 5.4. \tag{12}$$

$$\frac{\chi_{\rm exp}}{\chi_{\rm FG}} = \frac{m_3^*}{m_3}\Big(1+\frac{Z_0}{4}\Big)^{-1} = 9.2 \qquad \rightarrow \qquad Z_0 = -2.8. \tag{13}$$

iii) The comparison of the observed values of the thermal conductivity K and viscosity η, at 2 mK and atmospheric pressure, with those of other familiar liquids is given in the following table.

	^{3}He (0.002 K)	^{4}He (4.2 K)	N_2 (77 K)	Water (300 K)	Oil (300 K)
K (μW/m)	0.25	0.03	0.14	0.6	20
η ($\times10^{-3}$ poise)	625	0.036	1.5†	10	100–1000†

†These values are strongly temperature dependent because of the closeness to the liquid-solid phase transition. It is interesting that although it is far from solidification, the liquid ^{3}He at 0.002 K is more viscous than many oils.