Unkonventionelle Supraleitung WS 05/06
Lösungen zur Serie 3

3.1 The density operator is

$$\begin{aligned}
\rho_s(\mathbf{r}) &= \psi_s^\dagger(\mathbf{r})\psi_s(\mathbf{r}) && (1)\\
&= \Big(\frac{1}{\sqrt{V}}\sum_k e^{i\mathbf{k}\cdot\mathbf{r}}c_{k,s}^\dagger\Big)\Big(\frac{1}{\sqrt{V}}\sum_{k'} e^{-i\mathbf{k}'\cdot\mathbf{r}}c_{k',s}\Big) && (2)\\
&= \frac{1}{V}\sum_{k,k'} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}c_{k,s}^\dagger c_{k',s} && (3)\\
&= \frac{1}{V}\sum_{k,q} e^{i\mathbf{q}\cdot\mathbf{r}}c_{k,s}^\dagger c_{k-q,s}. \qquad (\mathbf{q}=\mathbf{k}-\mathbf{k}') && (4)
\end{aligned}$$

Let us consider H_{INT}.

$$\begin{aligned}
H_{\mathrm{INT}} &= \int d\mathbf{r}d\mathbf{r}' U\delta^3(\mathbf{r}-\mathbf{r}')\rho_\uparrow(\mathbf{r})\rho_\downarrow(\mathbf{r}') && (5)\\
&= U\int d\mathbf{r}\rho_\uparrow(\mathbf{r})\rho_\downarrow(\mathbf{r}) && (6)\\
&= U\int d\mathbf{r}\Big(\frac{1}{V}\sum_{k,q} e^{i\mathbf{q}\cdot\mathbf{r}}c_{k,\uparrow}^\dagger c_{k-q,\uparrow}\Big)\Big(\frac{1}{V}\sum_{k',q'} e^{i\mathbf{q}'\cdot\mathbf{r}}c_{k',\downarrow}^\dagger c_{k'-q',\downarrow}\Big) && (7)\\
&= \frac{U}{V}\sum_{k,q}\sum_{k',q'}\Big(\int\frac{d\mathbf{r}}{V}e^{i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{r}}\Big)c_{k,\uparrow}^\dagger c_{k-q,\uparrow}c_{k',\downarrow}^\dagger c_{k'-q',\downarrow} && (8)\\
&= I\sum_{k,k',q} c_{k,\uparrow}^\dagger c_{k-q,\uparrow}c_{k',\downarrow}^\dagger c_{k'+q,\downarrow}, && (9)
\end{aligned}$$

where $I \equiv U/V$. We define $b_{k,q,s}$ and $\delta b_{k,q,s}$ as

$$\begin{aligned}
b_{k,q,s} &= c_{k,s}^\dagger c_{k-q,s}, && (10)\\
\delta b_{k,q,s} &= c_{k,s}^\dagger c_{k-q,s} - \langle c_{k,s}^\dagger c_{k-q,s}\rangle = b_{k,q,s} - \langle b_{k,q,s}\rangle. && (11)
\end{aligned}$$

Substituting $b_{k,q,s} = \langle b_{k,q,s}\rangle + \delta b_{k,q,s}$ into H_{INT},

$$\begin{aligned}
H_{\mathrm{INT}} &= I\sum_{k,k',q} b_{k,q,\uparrow}b_{k',-q,\downarrow} && (12)\\
&= I\sum_{k,k',q}\Big(\langle b_{k,q,\uparrow}\rangle + \delta b_{k,q,\uparrow}\Big)\Big(\langle b_{k',-q,\downarrow}\rangle + \delta b_{k',-q,\downarrow}\Big) && (13)\\
&= I\sum_{k,k',q}\Big(\langle b_{k,q,\uparrow}\rangle\langle b_{k',-q,\downarrow}\rangle + \langle b_{k,q,\uparrow}\rangle\delta b_{k',-q,\downarrow} + \delta b_{k,q,\uparrow}\langle b_{k',-q,\downarrow}\rangle + \delta b_{k,q,\uparrow}\delta b_{k',-q,\downarrow}\Big) && (14)\\
&\approx I\sum_{k,k',q}\Big(\langle b_{k,q,\uparrow}\rangle\langle b_{k',-q,\downarrow}\rangle + \langle b_{k,q,\uparrow}\rangle\delta b_{k',-q,\downarrow} + \delta b_{k,q,\uparrow}\langle b_{k',-q,\downarrow}\rangle\Big) && (15)\\
&= I\sum_{k,k',q}\Big[\langle b_{k,q,\uparrow}\rangle\langle b_{k',-q,\downarrow}\rangle \\
&\qquad + \langle b_{k,q,\uparrow}\rangle\Big(b_{k',-q,\downarrow} - \langle b_{k',-q,\downarrow}\rangle\Big) + \Big(b_{k,q,\uparrow} - \langle b_{k,q,\uparrow}\rangle\Big)\langle b_{k',-q,\downarrow}\rangle\Big] && (16)\\
&= I\sum_{k,k',q}\Big[\langle b_{k,q,\uparrow}\rangle b_{k',-q,\downarrow} + b_{k,q,\uparrow}\langle b_{k',-q,\downarrow}\rangle - \langle b_{k,q,\uparrow}\rangle\langle b_{k',-q,\downarrow}\rangle\Big], && (17)
\end{aligned}$$

where we have assumed $\delta b_{k,\pm q,s}$ are small, and have neglected the quadratic term of them in Eq. (15) (the mean-field approximation).

Now, we are interested in a spatially-uniform spin polarization here, and then we assume $\langle b_{k,\pm q,s}\rangle = \langle c^\dagger_{k,s} c_{k\mp q,s}\rangle = 0$ for $\mathbf{q} \neq 0$. In this case, $H_{\rm INT}$ is

$$
\begin{aligned}
H_{\rm INT} &= I\sum_{k,k'}\Big[\langle b_{k,q=0,\uparrow}\rangle b_{k',-q=0,\downarrow} + b_{k,q=0,\uparrow}\langle b_{k',-q=0,\downarrow}\rangle - \langle b_{k,q=0,\uparrow}\rangle\langle b_{k',-q=0,\downarrow}\rangle\Big] & (18)\\
&= I\sum_{k,k'}\Big[\langle c^\dagger_{k,\uparrow}c_{k,\uparrow}\rangle c^\dagger_{k',\downarrow}c_{k',\downarrow} + c^\dagger_{k,\uparrow}c_{k,\uparrow}\langle c^\dagger_{k',\downarrow}c_{k',\downarrow}\rangle - \langle c^\dagger_{k,\uparrow}c_{k,\uparrow}\rangle\langle c^\dagger_{k',\downarrow}c_{k',\downarrow}\rangle\Big] & (19)\\
&= I\sum_{k}(N_\uparrow n_{k,\downarrow} + n_{k,\uparrow}N_\downarrow) - IN_\uparrow N_\downarrow, & (20)
\end{aligned}
$$

where

$$
\begin{aligned}
N_s &\equiv \sum_k \langle c^\dagger_{k,s}c_{k,s}\rangle = \sum_k \langle n_{k,s}\rangle, & (21)\\
n_{k,s} &\equiv c^\dagger_{k,s}c_{k,s}. & (22)
\end{aligned}
$$

Hence, the total Hamiltonian H_1 is

$$
\begin{aligned}
H_1 &= H_{\rm KE} + H_{\rm INT} + H_{\rm Z}^{(A)} & (23)\\
&= \sum_k \xi_k(n_{k,\uparrow} + n_{k,\downarrow}) + \Big[I\sum_k(N_\uparrow n_{k,\downarrow} + n_{k,\uparrow}N_\downarrow) - IN_\uparrow N_\downarrow\Big] - \mu_{\rm B}H\sum_k(n_{k,\uparrow} - n_{k,\downarrow}) & (24)\\
&= \sum_k\Big[(\xi_k + IN_\downarrow - \mu_{\rm B}H)n_{k,\uparrow} + (\xi_k + IN_\uparrow + \mu_{\rm B}H)n_{k,\downarrow}\Big] - IN_\uparrow N_\downarrow & (25)\\
&= \sum_k\Big[\tilde{\xi}_{k,\uparrow}n_{k,\uparrow} + \tilde{\xi}_{k,\downarrow}n_{k,\downarrow}\Big] - IN_\uparrow N_\downarrow, & (26)
\end{aligned}
$$

where

$$
\tilde{\xi}_{k,\uparrow} \equiv \xi_k + IN_\downarrow - \mu_{\rm B}H, \qquad \tilde{\xi}_{k,\downarrow} \equiv \xi_k + IN_\uparrow + \mu_{\rm B}H. \tag{27}
$$

This means that the original spin-degenerate band ξ_k is split into two ones, $\tilde{\xi}_{k,\uparrow}$ and $\tilde{\xi}_{k,\downarrow}$. As a result,

$$
\begin{aligned}
\langle n_{k,\uparrow}\rangle &= f(\tilde{\xi}_{k,\uparrow}) & (28)\\
&= f(\xi_k + IN_\downarrow - \mu_{\rm B}H), & (29)
\end{aligned}
$$

and

$$
\begin{aligned}
\langle n_{k,\downarrow}\rangle &= f(\tilde{\xi}_{k,\downarrow}) & (30)\\
&= f(\xi_k + IN_\uparrow + \mu_{\rm B}H). & (31)
\end{aligned}
$$

Here, $f(\xi)$ is the Fermi distribution function.

Next, the magnetization M and the total number of the electrons N are given as

$$
M = \mu_{\rm B}(N_\uparrow - N_\downarrow), \qquad N = N_\uparrow + N_\downarrow, \tag{32}
$$

and therefore

$$
N_\uparrow = \frac{1}{2}\Big(N + \frac{M}{\mu_{\rm B}}\Big), \qquad N_\downarrow = \frac{1}{2}\Big(N - \frac{M}{\mu_{\rm B}}\Big). \tag{33}
$$

Substituting these $N_\uparrow$ and $N_\downarrow$ into Eq. (25),

$$
\begin{aligned}
H_1 &= \sum_k \Big[\Big\{ \xi_k + I\frac{1}{2}\Big(N - \frac{M}{\mu_\mathrm{B}}\Big) - \mu_\mathrm{B} H \Big\} n_{k,\uparrow} + \Big\{ \xi_k + I\frac{1}{2}\Big(N + \frac{M}{\mu_\mathrm{B}}\Big) + \mu_\mathrm{B} H \Big\} n_{k,\downarrow} \Big] \\
&\quad - I N_\uparrow N_\downarrow \qquad (34) \\
&= \sum_k \Big[\big\{ \xi_k + \frac{1}{2} I N \big\} n_{k,\uparrow} + \big\{ \xi_k + \frac{1}{2} I N \big\} n_{k,\downarrow} \\
&\qquad + \big\{ -\frac{1}{2}\frac{IM}{\mu_\mathrm{B}} - \mu_\mathrm{B} H \big\} n_{k,\uparrow} + \big\{ \frac{1}{2}\frac{IM}{\mu_\mathrm{B}} + \mu_\mathrm{B} H \big\} n_{k,\downarrow} \Big] - I N_\uparrow N_\downarrow \qquad (35) \\
&= \sum_{k,s} \Big(\xi_k + \frac{1}{2} I N \Big) n_{k,s} - \Big(\frac{1}{2}\frac{IM}{\mu_\mathrm{B}} + \mu_\mathrm{B} H \Big) \sum_k (n_{k,\uparrow} - n_{k,\downarrow}) - I N_\uparrow N_\downarrow \qquad (36) \\
&= \sum_{k,s} \Big(\xi_k + \frac{1}{2} I N \Big) n_{k,s} - \Big(\frac{1}{2}\frac{IM}{\mu_\mathrm{B}} + \mu_\mathrm{B} H \Big) \sum_k (n_{k,\uparrow} - n_{k,\downarrow}) - \frac{1}{4} I \Big(N^2 - \frac{M^2}{\mu_\mathrm{B}^2} \Big). \qquad (37)
\end{aligned}
$$

The energy E of the system is obtained as

$$
\begin{aligned}
E &= \langle H_1 \rangle \qquad (38) \\
&= \sum_{k,s} \Big(\xi_k + \frac{1}{2} I N \Big) \langle n_{k,s} \rangle - \Big(\frac{1}{2}\frac{IM}{\mu_\mathrm{B}} + \mu_\mathrm{B} H \Big) \sum_k \Big(\langle n_{k,\uparrow} \rangle - \langle n_{k,\downarrow} \rangle \Big) - \frac{1}{4} I \Big(N^2 - \frac{M^2}{\mu_\mathrm{B}^2} \Big) \qquad (39) \\
&= \sum_k \xi_k \Big[\langle n_{k,\uparrow} \rangle + \langle n_{k,\downarrow} \rangle \Big] + \frac{1}{2} I N (N_\uparrow + N_\downarrow) \\
&\quad - \Big(\frac{1}{2}\frac{IM}{\mu_\mathrm{B}} + \mu_\mathrm{B} H \Big) (N_\uparrow - N_\downarrow) - \frac{1}{4} I \Big(N^2 - \frac{M^2}{\mu_\mathrm{B}^2} \Big) \qquad (40) \\
&= \sum_k \xi_k \Big[f(\tilde{\xi}_{k,\uparrow}) + f(\tilde{\xi}_{k,\downarrow}) \Big] + \frac{1}{2} I N^2 \\
&\quad - \Big(\frac{1}{2}\frac{IM}{\mu_\mathrm{B}} + \mu_\mathrm{B} H \Big) \Big(\frac{M}{\mu_\mathrm{B}} \Big) - \frac{1}{4} I \Big(N^2 - \frac{M^2}{\mu_\mathrm{B}^2} \Big) \qquad (41) \\
&= \sum_k \xi_k \Big[f(\tilde{\xi}_{k,\uparrow}) + f(\tilde{\xi}_{k,\downarrow}) \Big] + \frac{1}{2} I N^2 + \Big(-\frac{1}{2}\frac{IM^2}{\mu_\mathrm{B}^2} - HM \Big) + \Big(-\frac{1}{4} I N^2 + \frac{1}{4}\frac{IM^2}{\mu_\mathrm{B}^2} \Big) \qquad (42) \\
&= \sum_k \xi_k \Big[f(\tilde{\xi}_{k,\uparrow}) + f(\tilde{\xi}_{k,\downarrow}) \Big] + \frac{1}{4} I N^2 - \frac{1}{4}\frac{IM^2}{\mu_\mathrm{B}^2} - HM \qquad (43)
\end{aligned}
$$

Here,

$$
\begin{aligned}
\sum_k \xi_k f(\tilde{\xi}_{k,\uparrow}) &= \sum_k \xi_k f(\xi_k + I N_\downarrow - \mu_\mathrm{B} H) \qquad (44) \\
&= \sum_k \xi_k f\Big(\xi_k + IN/2 - IM/2\mu_\mathrm{B} - \mu_\mathrm{B} H \Big) \qquad (45) \\
&= \sum_k \xi_k f\Big(\xi_k + IN/2 - I\chi H/2\mu_\mathrm{B} - \mu_\mathrm{B} H \Big), \qquad (46)
\end{aligned}
$$

$$
\begin{aligned}
\sum_k \xi_k f(\tilde{\xi}_{k,\downarrow}) &= \sum_k \xi_k f(\xi_k + I N_\uparrow + \mu_\mathrm{B} H) \qquad (47) \\
&= \sum_k \xi_k f\Big(\xi_k + IN/2 + IM/2\mu_\mathrm{B} + \mu_\mathrm{B} H \Big) \qquad (48) \\
&= \sum_k \xi_k f\Big(\xi_k + IN/2 + I\chi H/2\mu_\mathrm{B} + \mu_\mathrm{B} H \Big), \qquad (49)
\end{aligned}
$$

and therefore, for sufficiently small H,

$$\begin{aligned}
\sum_k \xi_k\Big[f(\tilde{\xi}_{k,\uparrow})+f(\tilde{\xi}_{k,\downarrow})\Big] &= \sum_k \xi_k\Big[f(\xi_k+IN/2-I\chi H/2\mu_{\rm B}-\mu_{\rm B}H) \\
&\qquad + f(\xi_k+IN/2+I\chi H/2\mu_{\rm B}+\mu_{\rm B}H)\Big] \qquad (50)\\
&\approx \sum_k \xi_k\Big[f(\xi_k+IN/2)-\beta f'(\xi_k+IN/2)\Big(I\chi H/2\mu_{\rm B}+\mu_{\rm B}H\Big) \\
&\qquad +\frac{\beta^2}{2}f''(\xi_k+IN/2)\Big(I\chi H/2\mu_{\rm B}+\mu_{\rm B}H\Big)^2 \\
&\quad + f(\xi_k+IN/2)+\beta f'(\xi_k+IN/2)\Big(I\chi H/2\mu_{\rm B}+\mu_{\rm B}H\Big) \\
&\qquad +\frac{\beta^2}{2}f''(\xi_k+IN/2)\Big(I\chi H/2\mu_{\rm B}+\mu_{\rm B}H\Big)^2\Big] \qquad (51)\\
&= 2\sum_k \xi_k f(\xi_k+IN/2) \\
&\quad +\beta^2\Big(I\chi H/2\mu_{\rm B}+\mu_{\rm B}H\Big)^2\sum_k \xi_k f''(\xi_k+IN/2), \qquad (52)
\end{aligned}$$

where $\beta=1/k_{\rm B}T$. As to the second term in the last line,

$$\begin{aligned}
\sum_k \xi_k f''(\xi_k+IN/2) &= \sum_k(\bar{\xi}_k-IN/2)f''(\bar{\xi}_k), \qquad (\bar{\xi}_k\equiv\xi_k+IN/2) \qquad (53)\\
&= \int_{-\infty}^{\infty}d\bar{\xi}_k D(\bar{\xi}_k)(\bar{\xi}_k-IN/2)f''(\bar{\xi}_k) \qquad (54)\\
&= \frac{1}{\beta}\Big[\Big\{D(\bar{\xi}_k)(\bar{\xi}_k-IN/2)\Big\}f'(\bar{\xi}_k)\Big]_{-\infty}^{\infty} \\
&\quad -\frac{1}{\beta}\int_{-\infty}^{\infty}d\bar{\xi}_k\frac{d}{d\bar{\xi}_k}\Big\{D(\bar{\xi}_k)(\bar{\xi}_k-IN/2)\Big\}f'(\bar{\xi}_k) \qquad (55)\\
&= \frac{1}{\beta}\int_{-\infty}^{\infty}d\bar{\xi}_k\Big\{D(\bar{\xi}_k)+D'(\bar{\xi}_k)(\bar{\xi}_k-IN/2)\Big\}\Big\{-f'(\bar{\xi}_k)\Big\} \qquad (56)\\
&\approx \frac{1}{\beta^2}\Big\{D(0)+D'(0)(-IN/2)\Big\} \qquad (57)\\
&\approx \frac{1}{\beta^2}D(0) \quad \equiv \frac{1}{\beta^2}D_{\rm F}, \qquad (58)
\end{aligned}$$

where $df(\xi)/d\xi=\beta f'(\xi)$, at sufficiently low temperatures $-f'(\xi)\approx\delta(\xi)/\beta$, and we have assumed that the derivative of the density of states at the Fermi level $D'(0)$ is small such that $D(0)\gg|D'(0)IN/2|$. Hence,

$$\begin{aligned}
E &= \sum_k \xi_k\Big[f(\tilde{\xi}_{k,\uparrow})+f(\tilde{\xi}_{k,\downarrow})\Big]+\frac{1}{4}IN^2-\frac{1}{4}\frac{IM^2}{\mu_{\rm B}^2}-HM \qquad (59)\\
&= 2\sum_k \xi_k f(\xi_k+IN/2) \\
&\quad + D_{\rm F}\Big(I\chi H/2\mu_{\rm B}+\mu_{\rm B}H\Big)^2+\frac{1}{4}IN^2-\frac{1}{4}\frac{IM^2}{\mu_{\rm B}^2}-HM \qquad (60)\\
&= 2\sum_k \xi_k f(\xi_k+IN/2) \\
&\quad + D_{\rm F}\Big(\frac{I\chi}{2\mu_{\rm B}}+\mu_{\rm B}\Big)^2H^2+\frac{1}{4}IN^2-\frac{I\chi^2H^2}{4\mu_{\rm B}^2}-\chi H^2 \qquad (61)
\end{aligned}$$

$$= \quad E_0 + \Big[D_{\rm F} \Big(\frac{I\chi}{2\mu_{\rm B}} + \mu_{\rm B} \Big)^2 - \frac{I\chi^2}{4\mu_{\rm B}^2} - \chi \Big] H^2, \tag{62}$$

where E_0 is the H-independent contribution to E,

$$E_0 \equiv 2 \sum_k \xi_k f(\xi_k + IN/2) + \frac{1}{4} IN^2. \tag{63}$$

Now, let us calculate the magnetization M.

$$M \quad = \quad -\frac{dE}{dH} \tag{64}$$

$$= \quad -2\Big[D_{\rm F} \Big(\frac{I\chi}{2\mu_{\rm B}} + \mu_{\rm B} \Big)^2 - \frac{I\chi^2}{4\mu_{\rm B}^2} - \chi \Big] H \tag{65}$$

$$= \quad -2\Big[D_{\rm F} \Big(\frac{I^2\chi^2}{4\mu_{\rm B}^2} + 2\mu_{\rm B} \frac{I\chi}{2\mu_{\rm B}} + \mu_{\rm B}^2 \Big) - \frac{I\chi^2}{4\mu_{\rm B}^2} - \chi \Big] H \tag{66}$$

$$= \quad 2\Big[-ID_{\rm F} \frac{I\chi^2}{4\mu_{\rm B}^2} - ID_{\rm F}\chi - \mu_{\rm B}^2 D_{\rm F} + \frac{I\chi^2}{4\mu_{\rm B}^2} + \chi \Big] H \tag{67}$$

$$= \quad 2\Big[(1 - ID_{\rm F}) \frac{I\chi^2}{4\mu_{\rm B}^2} + (1 - ID_{\rm F})\chi - \mu_{\rm B}^2 D_{\rm F} \Big] H. \tag{68}$$

From $M = \chi H$ and the above equation,

$$\chi \quad = \quad 2\Big[(1 - ID_{\rm F}) \frac{I\chi^2}{4\mu_{\rm B}^2} + (1 - ID_{\rm F})\chi - \mu_{\rm B}^2 D_{\rm F} \Big] \tag{69}$$

$$\rightarrow \qquad 0 \quad = \quad 2(1 - ID_{\rm F}) \frac{I\chi^2}{4\mu_{\rm B}^2} + (1 - 2ID_{\rm F})\chi - 2\mu_{\rm B}^2 D_{\rm F} \tag{70}$$

$$\rightarrow \qquad 0 \quad = \quad (1 - ID_{\rm F}) I\chi^2 + 2\mu_{\rm B}^2 (1 - 2ID_{\rm F})\chi - (2\mu_{\rm B}^2)^2 D_{\rm F}. \tag{71}$$

Then,

$$\chi \quad = \quad \frac{1}{(1 - ID_{\rm F})I} \Big[-\mu_{\rm B}^2 (1 - 2ID_{\rm F}) \pm \sqrt{(\mu_{\rm B}^2)^2 (1 - 2ID_{\rm F})^2 + (1 - ID_{\rm F}) I (2\mu_{\rm B}^2)^2 D_{\rm F}} \Big] \tag{72}$$

$$= \quad \frac{1}{(1 - ID_{\rm F})I} \Big[-\mu_{\rm B}^2 (1 - 2ID_{\rm F}) \pm \sqrt{(\mu_{\rm B}^2)^2 (1 - 4ID_{\rm F} + 4I^2 D_{\rm F}^2) + (\mu_{\rm B}^2)^2 (4ID_{\rm F} - 4I^2 D_{\rm F}^2)} \Big] \tag{73}$$

$$= \quad \frac{1}{(1 - ID_{\rm F})I} \Big[-\mu_{\rm B}^2 (1 - 2ID_{\rm F}) \pm \mu_{\rm B}^2 \Big] \tag{74}$$

$$= \quad \begin{cases} \frac{2\mu_{\rm B}^2 D_{\rm F}}{1 - ID_{\rm F}} \\ \frac{-2\mu_{\rm B}^2}{I} \end{cases} \tag{75}$$

The solution $\chi = -2\mu_{\rm B}^2/I$ is physically irrelevant because $\chi \rightarrow \infty$ for $I \rightarrow 0$. Therefore, we adopt

$$\chi \quad = \quad \frac{2\mu_{\rm B}^2 D_{\rm F}}{1 - ID_{\rm F}}, \tag{76}$$

which gives $\chi \rightarrow 2\mu_{\rm B}^2 D_{\rm F} = \chi_{\rm pauli}$ for $I \rightarrow 0$.

3.2 The x component of the spin density operator is

$$
\begin{aligned}
S_x(\mathbf{r}) &= \sum_{s,s'} \psi_s^\dagger(\mathbf{r})[\sigma_x]_{ss'}\psi_{s'}(\mathbf{r}) &(77)\\
&= \psi_\uparrow^\dagger(\mathbf{r})\psi_\downarrow(\mathbf{r}) + \psi_\downarrow^\dagger(\mathbf{r})\psi_\uparrow(\mathbf{r}) &(78)\\
&= \Big(\frac{1}{\sqrt{V}}\sum_k e^{i\mathbf{k}\cdot\mathbf{r}} c_{k,\uparrow}^\dagger\Big)\Big(\frac{1}{\sqrt{V}}\sum_{k'} e^{-i\mathbf{k}'\cdot\mathbf{r}} c_{k',\downarrow}\Big)\\
&\quad + \Big(\frac{1}{\sqrt{V}}\sum_k e^{i\mathbf{k}\cdot\mathbf{r}} c_{k,\downarrow}^\dagger\Big)\Big(\frac{1}{\sqrt{V}}\sum_{k'} e^{-i\mathbf{k}'\cdot\mathbf{r}} c_{k',\uparrow}\Big) &(79)\\
&= \frac{1}{V}\sum_{k,k'} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} c_{k,\uparrow}^\dagger c_{k',\downarrow} + \frac{1}{V}\sum_{k,k'} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} c_{k,\downarrow}^\dagger c_{k',\uparrow} &(80)\\
&= \frac{1}{V}\sum_{k,q'} e^{i\mathbf{q}'\cdot\mathbf{r}}\Big(c_{k,\uparrow}^\dagger c_{k-q',\downarrow} + c_{k,\downarrow}^\dagger c_{k-q',\uparrow}\Big). \qquad (\mathbf{q}' = \mathbf{k}-\mathbf{k}') &(81)
\end{aligned}
$$

When $\vec{H}_q = H_q\hat{x}$,

$$
\begin{aligned}
H_{\rm Z}^{(B)} &= -V\sum_{i=1}^N \mu_{\rm B}\vec{S}(\mathbf{r}_i)\cdot\vec{H}_q\sqrt{2}\cos(\mathbf{q}\cdot\mathbf{r}_i) &(82)\\
&= -V\sum_{i=1}^N \mu_{\rm B}S_x(\mathbf{r}_i)H_q\sqrt{2}\cos(\mathbf{q}\cdot\mathbf{r}_i) &(83)\\
&= -V\sum_{i=1}^N \mu_{\rm B}\bigg(\frac{1}{V}\sum_{k,q'} e^{i\mathbf{q}'\cdot\mathbf{r}_i}\Big(c_{k,\uparrow}^\dagger c_{k-q',\downarrow} + c_{k,\downarrow}^\dagger c_{k-q',\uparrow}\Big)\bigg)H_q\sqrt{2}\cos(\mathbf{q}\cdot\mathbf{r}_i) &(84)\\
&= -\mu_{\rm B}H_q\sum_{k,q'}\Big(c_{k,\uparrow}^\dagger c_{k-q',\downarrow} + c_{k,\downarrow}^\dagger c_{k-q',\uparrow}\Big)\sum_{i=1}^N e^{i\mathbf{q}'\cdot\mathbf{r}_i}\sqrt{2}\cos(\mathbf{q}\cdot\mathbf{r}_i) &(85)\\
&= -\mu_{\rm B}H_q\sum_{k,q'}\Big(c_{k,\uparrow}^\dagger c_{k-q',\downarrow} + c_{k,\downarrow}^\dagger c_{k-q',\uparrow}\Big)\sum_{i=1}^N e^{i\mathbf{q}'\cdot\mathbf{r}_i}\sqrt{2}\frac{e^{i\mathbf{q}\cdot\mathbf{r}_i} + e^{-i\mathbf{q}\cdot\mathbf{r}_i}}{2} &(86)\\
&= -\frac{\mu_{\rm B}H_q}{\sqrt{2}}\sum_{k,q'}\Big(c_{k,\uparrow}^\dagger c_{k-q',\downarrow} + c_{k,\downarrow}^\dagger c_{k-q',\uparrow}\Big)\sum_{i=1}^N\Big(e^{i(\mathbf{q}'+\mathbf{q})\cdot\mathbf{r}_i} + e^{i(\mathbf{q}'-\mathbf{q})\cdot\mathbf{r}_i}\Big). &(87)
\end{aligned}
$$

Here, we assume that the electrons are distributed uniformly in the system with the volume V such that

$$
\sum_{i=1}^N \quad\rightarrow\quad \frac{1}{V}\int_V d\mathbf{r}_i. \tag{88}
$$

Then,

$$
\begin{aligned}
H_{\rm Z}^{(B)} &= -\frac{\mu_{\rm B}H_q}{\sqrt{2}}\sum_{k,q'}\Big(c_{k,\uparrow}^\dagger c_{k-q',\downarrow} + c_{k,\downarrow}^\dagger c_{k-q',\uparrow}\Big)\sum_{i=1}^N\Big(e^{i(\mathbf{q}'+\mathbf{q})\cdot\mathbf{r}_i} + e^{i(\mathbf{q}'-\mathbf{q})\cdot\mathbf{r}_i}\Big) &(89)\\
&= -\frac{\mu_{\rm B}H_q}{\sqrt{2}}\sum_{k,q'}\Big(c_{k,\uparrow}^\dagger c_{k-q',\downarrow} + c_{k,\downarrow}^\dagger c_{k-q',\uparrow}\Big)\frac{1}{V}\int_V d\mathbf{r}_i\Big(e^{i(\mathbf{q}'+\mathbf{q})\cdot\mathbf{r}_i} + e^{i(\mathbf{q}'-\mathbf{q})\cdot\mathbf{r}_i}\Big) &(90)\\
&= -\frac{\mu_{\rm B}H_q}{\sqrt{2}}\sum_k\Big[\Big(c_{k,\uparrow}^\dagger c_{k+q,\downarrow} + c_{k,\downarrow}^\dagger c_{k+q,\uparrow}\Big) + \Big(c_{k,\uparrow}^\dagger c_{k-q,\downarrow} + c_{k,\downarrow}^\dagger c_{k-q,\uparrow}\Big)\Big]. &(91)
\end{aligned}
$$

Considering $H_{\rm Z}^{(B)}$ as a perturbation term, and we calculate the energies $\tilde{\xi}_{k,s}$ of the single particle states up to the 2nd order correction by the perturbation theory of the quantum mechanics. Here, the unperturbed Hamiltonian H_0 is perturbed by $H_{\rm Z}^{(B)}$, resulting in $\tilde{H}$:

$$\begin{aligned} H_0 &\equiv H_{\rm KE} = \sum_{k,s} \xi_k c^\dagger_{k,s} c_{k,s} \\ \to \qquad \tilde{H} &= \sum_{k,s} \tilde{\xi}_{k,s} \tilde{c}^\dagger_{k,s} \tilde{c}_{k,s}. \end{aligned}$$

Defining $|\phi_{k,s}\rangle = c^\dagger_{k,s}|0\rangle$, ($|0\rangle$ is the vacuum state),

$$\tilde{\xi}_{k,s} = \xi_k + \langle \phi_{k,s}|H_{\rm Z}^{(B)}|\phi_{k,s}\rangle - {\sum_{k',s'}}' \frac{\langle \phi_{k,s}|H_{\rm Z}^{(B)}|\phi_{k',s'}\rangle\langle \phi_{k',s'}|H_{\rm Z}^{(B)}|\phi_{k,s}\rangle}{\xi_{k'} - \xi_k} \tag{92}$$

$$\begin{aligned} &= \xi_k - \frac{\mu_{\rm B} H_q}{\sqrt{2}} \sum_{k'} \Big\langle \phi_{k,s} \Big| [(c^\dagger_{k',\uparrow} c_{k'+q,\downarrow} + c^\dagger_{k',\downarrow} c_{k'+q,\uparrow}) + (c^\dagger_{k',\uparrow} c_{k'-q,\downarrow} + c^\dagger_{k',\downarrow} c_{k'-q,\uparrow})] \Big| \phi_{k,s} \Big\rangle \\ &\quad - \Big(\frac{\mu_{\rm B} H_q}{\sqrt{2}}\Big)^2 {\sum_{k',s'}}' \frac{1}{\xi_{k'} - \xi_k} \\ &\quad \times \left| \Big\langle \phi_{k,s} \Big| \sum_{k_1} [(c^\dagger_{k_1,\uparrow} c_{k_1+q,\downarrow} + c^\dagger_{k_1,\downarrow} c_{k_1+q,\uparrow}) + (c^\dagger_{k_1,\uparrow} c_{k_1-q,\downarrow} + c^\dagger_{k_1,\downarrow} c_{k_1-q,\uparrow})] \Big| \phi_{k',s'} \Big\rangle \right|^2 \end{aligned} \tag{93}$$

$$\begin{aligned} &= \xi_k - 0 \\ &\quad - \Big(\frac{\mu_{\rm B} H_q}{\sqrt{2}}\Big)^2 {\sum_{k',s'}}' \frac{1}{\xi_{k'} - \xi_k} \\ &\quad \times \Bigg| \sum_{k_1} \Big[(\delta_{k,k_1}\delta_{s,\uparrow})(\delta_{k_1+q,k'}\delta_{\downarrow,s'}) + (\delta_{k,k_1}\delta_{s,\downarrow})(\delta_{k_1+q,k'}\delta_{\uparrow,s'}) \\ &\qquad + (\delta_{k,k_1}\delta_{s,\uparrow})(\delta_{k_1-q,k'}\delta_{\downarrow,s'}) + (\delta_{k,k_1}\delta_{s,\downarrow})(\delta_{k_1-q,k'}\delta_{\uparrow,s'}) \Big] \Bigg|^2 \end{aligned} \tag{94}$$

$$\begin{aligned} &= \xi_k - \Big(\frac{\mu_{\rm B} H_q}{\sqrt{2}}\Big)^2 {\sum_{k',s'}}' \frac{1}{\xi_{k'} - \xi_k} \\ &\qquad \times \Bigg| \Big[\delta_{s,\uparrow}\delta_{\downarrow,s'}\delta_{k+q,k'} + \delta_{s,\downarrow}\delta_{\uparrow,s'}\delta_{k+q,k'} \\ &\qquad\qquad + \delta_{s,\uparrow}\delta_{\downarrow,s'}\delta_{k-q,k'} + \delta_{s,\downarrow}\delta_{\uparrow,s'}\delta_{k-q,k'} \Big] \Bigg|^2 \end{aligned} \tag{95}$$

$$\begin{aligned} &= \xi_k - \Big(\frac{\mu_{\rm B} H_q}{\sqrt{2}}\Big)^2 \sum_{s'} \\ &\qquad \times \Bigg[\delta_{s,\uparrow}\delta_{\downarrow,s'} \frac{1}{\xi_{k+q} - \xi_k} + \delta_{s,\downarrow}\delta_{\uparrow,s'} \frac{1}{\xi_{k+q} - \xi_k} \\ &\qquad\quad + \delta_{s,\uparrow}\delta_{\downarrow,s'} \frac{1}{\xi_{k-q} - \xi_k} + \delta_{s,\downarrow}\delta_{\uparrow,s'} \frac{1}{\xi_{k-q} - \xi_k} \Bigg] \end{aligned} \tag{96}$$

$$\begin{aligned} &= \xi_k - \Big(\frac{\mu_{\rm B} H_q}{\sqrt{2}}\Big)^2 \\ &\qquad \times \Bigg[\delta_{s,\uparrow} \frac{1}{\xi_{k+q} - \xi_k} + \delta_{s,\downarrow} \frac{1}{\xi_{k+q} - \xi_k} + \delta_{s,\uparrow} \frac{1}{\xi_{k-q} - \xi_k} + \delta_{s,\downarrow} \frac{1}{\xi_{k-q} - \xi_k} \Bigg] \end{aligned} \tag{97}$$

$$= \xi_k - \Big(\frac{\mu_{\rm B} H_q}{\sqrt{2}}\Big)^2 \Bigg[\Big(\frac{1}{\xi_{k+q} - \xi_k} \Big) + \Big(\frac{1}{\xi_{k-q} - \xi_k} \Big) \Bigg] \tag{98}$$

$$= \quad \xi_k - \frac{\mu_{\rm B}^2 H_q^2}{2}\left(\frac{1}{\xi_{k+q}-\xi_k} + \frac{1}{\xi_{k-q}-\xi_k}\right). \tag{99}$$

Hence, the energy of the system E is

$$E \quad = \quad \langle \tilde{H} \rangle = \sum_{k,s} \tilde{\xi}_{k,s} \langle \tilde{c}_{k,s}^\dagger \tilde{c}_{k,s} \rangle \tag{100}$$

$$= \quad \sum_k \left[\xi_k - \frac{\mu_{\rm B}^2 H_q^2}{2}\left(\frac{1}{\xi_{k+q}-\xi_k} + \frac{1}{\xi_{k-q}-\xi_k}\right)\right]\Big(\langle \tilde{c}_{k,\uparrow}^\dagger \tilde{c}_{k,\uparrow}\rangle + \langle \tilde{c}_{k,\downarrow}^\dagger \tilde{c}_{k,\downarrow}\rangle\Big) \tag{101}$$

$$= \quad 2\sum_k \tilde{\xi}_k f(\tilde{\xi}_k), \tag{102}$$

where

$$\tilde{\xi}_k \equiv \tilde{\xi}_{k,s} = \xi_k - \frac{\mu_{\rm B}^2 H_q^2}{2}\left(\frac{1}{\xi_{k+q}-\xi_k} + \frac{1}{\xi_{k-q}-\xi_k}\right). \tag{103}$$

By the Taylor expansion,

$$f(\tilde{\xi}_k) \quad \approx \quad f(\xi_k) + \beta f'(\xi_k)\frac{-\mu_{\rm B}^2 H_q^2}{2}\left(\frac{1}{\xi_{k+q}-\xi_k} + \frac{1}{\xi_{k-q}-\xi_k}\right). \tag{104}$$

Thus,

$$E \quad = \quad 2\sum_k \tilde{\xi}_k f(\tilde{\xi}_k) \tag{105}$$

$$\approx \quad 2\sum_k \left[\xi_k - \frac{\mu_{\rm B}^2 H_q^2}{2}\left(\frac{1}{\xi_{k+q}-\xi_k} + \frac{1}{\xi_{k-q}-\xi_k}\right)\right] \times \left[f(\xi_k) + \beta f'(\xi_k)\frac{-\mu_{\rm B}^2 H_q^2}{2}\left(\frac{1}{\xi_{k+q}-\xi_k} + \frac{1}{\xi_{k-q}-\xi_k}\right)\right] \tag{106}$$

$$\approx \quad 2\sum_k \left[\xi_k f(\xi_k) - f(\xi_k)\frac{\mu_{\rm B}^2 H_q^2}{2}\left(\frac{1}{\xi_{k+q}-\xi_k} + \frac{1}{\xi_{k-q}-\xi_k}\right) + \xi_k \beta f'(\xi_k)\frac{-\mu_{\rm B}^2 H_q^2}{2}\left(\frac{1}{\xi_{k+q}-\xi_k} + \frac{1}{\xi_{k-q}-\xi_k}\right)\right]. \tag{107}$$

Here, the third term is

$$2\sum_k \xi_k \beta f'(\xi_k)\frac{-\mu_{\rm B}^2 H_q^2}{2}\left(\frac{1}{\xi_{k+q}-\xi_k} + \frac{1}{\xi_{k-q}-\xi_k}\right) \tag{108}$$

$$= \quad 2\int \frac{d\Omega_k}{4\pi}\int_{-\infty}^{\infty} d\xi_k D(\xi_k)\xi_k \beta f'(\xi_k)\frac{-\mu_{\rm B}^2 H_q^2}{2}\left(\frac{1}{\xi_{k+q}-\xi_k} + \frac{1}{\xi_{k-q}-\xi_k}\right) \tag{109}$$

$$= \quad 2\int \frac{d\Omega_k}{4\pi}\int_{-\infty}^{\infty} d\xi_k D(\xi_k)\xi_k \Big\{-\beta f'(\xi_k)\Big\}\frac{\mu_{\rm B}^2 H_q^2}{2}\left(\frac{1}{\xi_{k+q}-\xi_k} + \frac{1}{\xi_{k-q}-\xi_k}\right) \tag{110}$$

$$\approx \quad 2\int \frac{d\Omega_k}{4\pi}\int_{-\infty}^{\infty} d\xi_k D(\xi_k)\xi_k \Big\{\delta(\xi_k)\Big\}\frac{\mu_{\rm B}^2 H_q^2}{2}\left(\frac{1}{\xi_{k+q}-\xi_k} + \frac{1}{\xi_{k-q}-\xi_k}\right) \tag{111}$$

$$= \quad 0. \tag{112}$$

Therefore,

$$E \quad = \quad 2\sum_k \left[\xi_k f(\xi_k) - f(\xi_k)\frac{\mu_{\rm B}^2 H_q^2}{2}\left(\frac{1}{\xi_{k+q}-\xi_k} + \frac{1}{\xi_{k-q}-\xi_k}\right)\right]. \tag{113}$$

The magnetization M is

$$M = -\frac{dE}{dH_q} \quad (114)$$
$$= 2\sum_k f(\xi_k)\mu_{\mathrm{B}}^2 H_q \left(\frac{1}{\xi_{k+q}-\xi_k} + \frac{1}{\xi_{k-q}-\xi_k} \right) \quad (115)$$
$$= 2\mu_{\mathrm{B}}^2 H_q \left(\sum_k \frac{f(\xi_k)}{\xi_{k+q}-\xi_k} + \sum_k \frac{f(\xi_k)}{\xi_{k-q}-\xi_k} \right) \quad (116)$$
$$= 2\mu_{\mathrm{B}}^2 H_q \left(\sum_k \frac{f(\xi_k)}{\xi_{k+q}-\xi_k} + \sum_k \frac{f(\xi_{k+q})}{\xi_k-\xi_{k+q}} \right) \quad (117)$$
$$= 2\mu_{\mathrm{B}}^2 H_q \left(\sum_k \frac{f(\xi_k)}{\xi_{k+q}-\xi_k} - \sum_k \frac{f(\xi_{k+q})}{\xi_{k+q}-\xi_k} \right) \quad (118)$$
$$= 2\mu_{\mathrm{B}}^2 H_q \sum_k \frac{f(\xi_k)-f(\xi_{k+q})}{\xi_{k+q}-\xi_k}. \quad (119)$$

Then, the non-uniform susceptibility $\chi(\mathbf{q})$ is

$$\chi(\mathbf{q}) = \frac{M}{H_q} \quad (120)$$
$$= 2\mu_{\mathrm{B}}^2 \sum_k \frac{f(\xi_k)-f(\xi_{k+q})}{\xi_{k+q}-\xi_k}. \quad (121)$$

We assume $\xi_k = \hbar^2(|\mathbf{k}|^2 - k_{\mathrm{F}}^2)/2m$.

$$|\mathbf{k} \pm \mathbf{q}|^2 = |\mathbf{k}|^2 \pm 2\mathbf{k}\cdot\mathbf{q} + |\mathbf{q}|^2. \quad (122)$$

$$\xi_{k\pm q} - \xi_k = \frac{\hbar^2(|\mathbf{k}\pm\mathbf{q}|^2 - k_{\mathrm{F}}^2)}{2m} - \frac{\hbar^2(|\mathbf{k}|^2 - k_{\mathrm{F}}^2)}{2m} \quad (123)$$
$$= \pm\frac{\hbar^2\mathbf{k}\cdot\mathbf{q}}{m} + \frac{\hbar^2|\mathbf{q}|^2}{2m}. \quad (124)$$

At low temperatures,

$$f(\xi_k) \approx \Theta(k_{\mathrm{F}}^2 - |\mathbf{k}|^2) \quad (125)$$

where $\Theta(k)$ is the step function:

$$\Theta(k) = \begin{cases} 1 & (k > 0) \\ 0 & (k < 0) \end{cases} \quad (126)$$

Under these conditions,

$$\chi(\mathbf{q}) = \frac{M}{H_q} \quad (127)$$

$$= 2\mu_{\mathrm{B}}^2 \sum_k \frac{f(\xi_k) - f(\xi_{k+q})}{\xi_{k+q} - \xi_k} \tag{128}$$

$$= 2\mu_{\mathrm{B}}^2 \sum_k \left(\frac{f(\xi_k)}{\xi_{k+q} - \xi_k} + \frac{f(\xi_k)}{\xi_{k-q} - \xi_k} \right) \tag{129}$$

$$\approx 2\mu_{\mathrm{B}}^2 \sum_k \left(\frac{\Theta(k_{\mathrm{F}}^2 - |\mathbf{k}|^2)}{\frac{\hbar^2 \mathbf{k}\cdot\mathbf{q}}{m} + \frac{\hbar^2|\mathbf{q}|^2}{2m}} + \frac{\Theta(k_{\mathrm{F}}^2 - |\mathbf{k}|^2)}{-\frac{\hbar^2 \mathbf{k}\cdot\mathbf{q}}{m} + \frac{\hbar^2|\mathbf{q}|^2}{2m}} \right) \tag{130}$$

$$= 4m\mu_{\mathrm{B}}^2 \sum_k \left(\frac{\Theta(k_{\mathrm{F}}^2 - |\mathbf{k}|^2)}{2\hbar^2 \mathbf{k}\cdot\mathbf{q} + \hbar^2|\mathbf{q}|^2} + \frac{\Theta(k_{\mathrm{F}}^2 - |\mathbf{k}|^2)}{-2\hbar^2 \mathbf{k}\cdot\mathbf{q} + \hbar^2|\mathbf{q}|^2} \right) \tag{131}$$

$$= 4m\mu_{\mathrm{B}}^2 \frac{V}{(2\pi)^3} \int d^3k \left(\frac{\Theta(k_{\mathrm{F}}^2 - |\mathbf{k}|^2)}{2\hbar^2 \mathbf{k}\cdot\mathbf{q} + \hbar^2|\mathbf{q}|^2} + \frac{\Theta(k_{\mathrm{F}}^2 - |\mathbf{k}|^2)}{-2\hbar^2 \mathbf{k}\cdot\mathbf{q} + \hbar^2|\mathbf{q}|^2} \right) \tag{132}$$

$$= 4m\mu_{\mathrm{B}}^2 \frac{V}{(2\pi)^3} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{\infty} dk k^2 \sin\theta \times \left(\frac{\Theta(k_{\mathrm{F}}^2 - k^2)}{2\hbar^2 kq\cos\theta + \hbar^2 q^2} + \frac{\Theta(k_{\mathrm{F}}^2 - k^2)}{-2\hbar^2 kq\cos\theta + \hbar^2 q^2} \right), \quad (q \equiv |\mathbf{q}|) \tag{133}$$

$$= 4m\mu_{\mathrm{B}}^2 \frac{V}{(2\pi)^3} (2\pi) \int_0^{\pi} d\theta \int_0^{\infty} dk k^2 \sin\theta \times \left(\frac{\Theta(k_{\mathrm{F}}^2 - k^2)}{2\hbar^2 kq\cos\theta + \hbar^2 q^2} + \frac{\Theta(k_{\mathrm{F}}^2 - k^2)}{-2\hbar^2 kq\cos\theta + \hbar^2 q^2} \right) \tag{134}$$

$$= \frac{m\mu_{\mathrm{B}}^2 V}{\pi^2\hbar^2} \int_0^{\pi} d\theta \int_0^{\infty} dk k^2 \sin\theta \left(\frac{\Theta(k_{\mathrm{F}}^2 - k^2)}{2kq\cos\theta + q^2} - \frac{\Theta(k_{\mathrm{F}}^2 - k^2)}{2kq\cos\theta - q^2} \right) \tag{135}$$

$$= \frac{m\mu_{\mathrm{B}}^2 V}{\pi^2\hbar^2} \int_0^{\pi} d\theta \int_0^{k_{\mathrm{F}}} dk k^2 \sin\theta \left(\frac{1}{2kq\cos\theta + q^2} - \frac{1}{2kq\cos\theta - q^2} \right) \tag{136}$$

$$= \frac{m\mu_{\mathrm{B}}^2 V}{\pi^2\hbar^2} \int_{-1}^{1} d(\cos\theta) \int_0^{k_{\mathrm{F}}} dk k^2 \left(\frac{1}{2kq\cos\theta + q^2} - \frac{1}{2kq\cos\theta - q^2} \right) \tag{137}$$

$$= \frac{m\mu_{\mathrm{B}}^2 V}{\pi^2\hbar^2 q} \int_{-1}^{1} dt \int_0^{k_{\mathrm{F}}} dk k^2 \left(\frac{1}{2kt + q} - \frac{1}{2kt - q} \right) \tag{138}$$

$$= \frac{m\mu_{\mathrm{B}}^2 V}{\pi^2\hbar^2 q} \int_{-1}^{1} dt \int_0^{k_{\mathrm{F}}} dk k^2 \frac{1}{2kt + q} - \frac{m\mu_{\mathrm{B}}^2 V}{\pi^2\hbar^2 q} \int_{-1}^{1} dt \int_0^{k_{\mathrm{F}}} dk k^2 \frac{1}{2kt - q} \tag{139}$$

$$= \frac{m\mu_{\mathrm{B}}^2 V}{\pi^2\hbar^2 q} \int_{-1}^{1} dt \left[\frac{-kq}{4t^2} + \frac{k^2}{4t} + \frac{q^2 \ln|2kt + q|}{8t^3} \right]_{k=0}^{k_{\mathrm{F}}} - \frac{m\mu_{\mathrm{B}}^2 V}{\pi^2\hbar^2 q} \int_{-1}^{1} dt \left[\frac{kq}{4t^2} + \frac{k^2}{4t} + \frac{q^2 \ln|2kt - q|}{8t^3} \right]_{k=0}^{k_{\mathrm{F}}} \tag{140}$$

$$= \frac{m\mu_{\mathrm{B}}^2 V}{\pi^2\hbar^2 q} \int_{-1}^{1} dt \left[\frac{-k_{\mathrm{F}}q}{4t^2} + \frac{k_{\mathrm{F}}^2}{4t} + \frac{q^2 \ln|2k_{\mathrm{F}}t + q|}{8t^3} \right] - \frac{m\mu_{\mathrm{B}}^2 V}{\pi^2\hbar^2 q} \int_{-1}^{1} dt \left[\frac{k_{\mathrm{F}}q}{4t^2} + \frac{k_{\mathrm{F}}^2}{4t} + \frac{q^2 \ln|2k_{\mathrm{F}}t - q|}{8t^3} \right] \tag{141}$$

$$= \frac{m\mu_{\mathrm{B}}^2 V}{\pi^2\hbar^2 q} \int_{-1}^{1} dt \left[\frac{-k_{\mathrm{F}}q}{2t^2} + \frac{q^2}{8t^3} \ln\frac{|2k_{\mathrm{F}}t + q|}{|2k_{\mathrm{F}}t - q|} \right] \tag{142}$$

$$= \frac{m\mu_{\rm B}^2 V}{\pi^2\hbar^2 q}\left[\frac{k_{\rm F}q}{2t} + \frac{q^2}{8}\Big(\frac{-2k_{\rm F}}{qt} + \frac{2k_{\rm F}^2}{q^2}\ln\frac{|2k_{\rm F}t+q|}{|2k_{\rm F}t-q|} - \frac{1}{2t^2}\ln\frac{|2k_{\rm F}t+q|}{|2k_{\rm F}t-q|}\Big)\right]_{t=-1}^{1} \quad (143)$$

$$= \frac{m\mu_{\rm B}^2 V}{\pi^2\hbar^2 q}\left[\frac{k_{\rm F}q}{2} + \frac{q^2}{8}\Big(\frac{-2k_{\rm F}}{q} + \frac{2k_{\rm F}^2}{q^2}\ln\frac{|2k_{\rm F}+q|}{|2k_{\rm F}-q|} - \frac{1}{2}\ln\frac{|2k_{\rm F}+q|}{|2k_{\rm F}-q|}\Big)\right]$$
$$- \frac{m\mu_{\rm B}^2 V}{\pi^2\hbar^2 q}\left[\frac{-k_{\rm F}q}{2} + \frac{q^2}{8}\Big(\frac{2k_{\rm F}}{q} + \frac{2k_{\rm F}^2}{q^2}\ln\frac{|2k_{\rm F}-q|}{|2k_{\rm F}+q|} - \frac{1}{2}\ln\frac{|2k_{\rm F}-q|}{|2k_{\rm F}+q|}\Big)\right] \quad (144)$$

$$= \frac{m\mu_{\rm B}^2 V}{\pi^2\hbar^2 q}\left[k_{\rm F}q + \frac{q^2}{8}\Big(\frac{-4k_{\rm F}}{q} + \frac{4k_{\rm F}^2}{q^2}\ln\frac{|2k_{\rm F}+q|}{|2k_{\rm F}-q|} - \ln\frac{|2k_{\rm F}+q|}{|2k_{\rm F}-q|}\Big)\right] \quad (145)$$

$$= \frac{m\mu_{\rm B}^2 V}{\pi^2\hbar^2 q}\left[k_{\rm F}q + \frac{-k_{\rm F}q}{2} + \frac{k_{\rm F}^2}{2}\ln\frac{|2k_{\rm F}+q|}{|2k_{\rm F}-q|} - \frac{q^2}{8}\ln\frac{|2k_{\rm F}+q|}{|2k_{\rm F}-q|}\right] \quad (146)$$

$$= \frac{m\mu_{\rm B}^2 V}{\pi^2\hbar^2 q}\left[\frac{k_{\rm F}q}{2} + \Big(\frac{k_{\rm F}^2}{2} - \frac{q^2}{8}\Big)\ln\frac{|2k_{\rm F}+q|}{|2k_{\rm F}-q|}\right] \quad (147)$$

$$= \frac{m\mu_{\rm B}^2 V}{\pi^2\hbar^2 q}\left[\frac{k_{\rm F}q}{2} + \frac{k_{\rm F}^2}{2}\Big(1 - \frac{q^2}{4k_{\rm F}^2}\Big)\ln\frac{|2k_{\rm F}+q|}{|2k_{\rm F}-q|}\right] \quad (148)$$

$$= \frac{m\mu_{\rm B}^2 V}{\pi^2\hbar^2}k_{\rm F}\left[\frac{1}{2} + \frac{k_{\rm F}}{2q}\Big(1 - \frac{q^2}{4k_{\rm F}^2}\Big)\ln\frac{|2k_{\rm F}+q|}{|2k_{\rm F}-q|}\right] \quad (149)$$

$$= \frac{2m}{\hbar^2 k_{\rm F}^2}\frac{\mu_{\rm B}^2 V}{2\pi^2}k_{\rm F}^3\left[\frac{1}{2} + \frac{k_{\rm F}}{2q}\Big(1 - \frac{q^2}{4k_{\rm F}^2}\Big)\ln\frac{|2k_{\rm F}+q|}{|2k_{\rm F}-q|}\right] \quad (150)$$

$$= \frac{1}{E_{\rm F}}\frac{\mu_{\rm B}^2 V}{2\pi^2}\Big(\frac{3\pi^2 N}{V}\Big)\left[\frac{1}{2} + \frac{k_{\rm F}}{2q}\Big(1 - \frac{q^2}{4k_{\rm F}^2}\Big)\ln\frac{|2k_{\rm F}+q|}{|2k_{\rm F}-q|}\right] \quad (151)$$

$$= 2\mu_{\rm B}^2\Big(\frac{3N}{4E_{\rm F}}\Big)\left[\frac{1}{2} + \frac{k_{\rm F}}{2q}\Big(1 - \frac{q^2}{4k_{\rm F}^2}\Big)\ln\frac{|2k_{\rm F}+q|}{|2k_{\rm F}-q|}\right] \quad (152)$$

$$= 2\mu_{\rm B}^2 D_{\rm F}\left[\frac{1}{2} + \frac{k_{\rm F}}{2q}\Big(1 - \frac{q^2}{4k_{\rm F}^2}\Big)\ln\frac{|2k_{\rm F}+q|}{|2k_{\rm F}-q|}\right]. \quad (153)$$

Let us assume $q \ll k_{\rm F}$. Then,

$$\ln|2k_{\rm F} \pm q| = \ln\Big|2k_{\rm F}\Big(1 \pm \frac{q}{2k_{\rm F}}\Big)\Big| \quad (154)$$

$$= \ln(2k_{\rm F}) + \ln\Big|1 \pm \frac{q}{2k_{\rm F}}\Big| \quad (155)$$

$$\approx \ln(2k_{\rm F}) \pm \frac{q}{2k_{\rm F}} - \frac{1}{2}\Big(\frac{q}{2k_{\rm F}}\Big)^2. \quad (156)$$

Thus,

$$\ln\frac{|2k_{\rm F}+q|}{|2k_{\rm F}-q|} = \ln\Big|1 + \frac{q}{2k_{\rm F}}\Big| - \ln\Big|1 - \frac{q}{2k_{\rm F}}\Big| \quad (157)$$

$$\approx \frac{q}{2k_{\rm F}} - \Big(-\frac{q}{2k_{\rm F}}\Big) \quad (158)$$

$$= \frac{q}{k_{\rm F}}. \quad (159)$$

Therefore, $(q = |\mathbf{q}|)$

$$\chi(\mathbf{q}) = 2\mu_{\rm B}^2 D_{\rm F}\left[\frac{1}{2} + \frac{k_{\rm F}}{2q}\Big(1 - \frac{q^2}{4k_{\rm F}^2}\Big)\ln\frac{|2k_{\rm F}+q|}{|2k_{\rm F}-q|}\right] \quad (160)$$

$$\approx \quad 2\mu_{\rm B}^2 D_{\rm F}\left[\frac{1}{2} + \frac{k_{\rm F}}{2q}\Big(1 - \frac{q^2}{4k_{\rm F}^2}\Big)\frac{q}{k_{\rm F}}\right] \tag{161}$$

$$= \quad 2\mu_{\rm B}^2 D_{\rm F}\left[\frac{1}{2} + \frac{1}{2}\Big(1 - \frac{q^2}{4k_{\rm F}^2}\Big)\right] \tag{162}$$

$$= \quad 2\mu_{\rm B}^2 D_{\rm F}\left[1 - \frac{q^2}{8k_{\rm F}^2}\right]. \tag{163}$$