Unkonventionelle Supraleitung WS 05/06
Lösungen zur Serie 4

4.1 The Hamiltonian is diagonalized, by the unitary (Bogoliubov) transformation $\check{U}_k$, as

$$\begin{aligned} H &= \sum_k \Big(C_k^\dagger \check{U}_k\Big)\Big(\check{U}_k^\dagger \check{\varepsilon}_k \check{U}_k\Big)\Big(\check{U}_k^\dagger C_k\Big) \\ &= \sum_k A_k^\dagger \check{E}_k A_k, \end{aligned} \tag{1}$$

where

$$\check{U}_k^\dagger \check{\varepsilon}_k \check{U}_k = \check{E}_k, \tag{2}$$

$$A_k = \check{U}_k^\dagger C_k. \tag{3}$$

From Eq. (2),

$$\check{\varepsilon}_k \check{U}_k = \check{U}_k \check{E}_k. \tag{4}$$

The left hand side of this is

$$\check{\varepsilon}_k \check{U}_k = \frac{1}{2}\begin{pmatrix} \xi_k \hat{\sigma}_0 & \hat{\Delta}_k \\ \hat{\Delta}_k^\dagger & -\xi_k \hat{\sigma}_0 \end{pmatrix}\begin{pmatrix} \hat{u}_k & \hat{v}_k \\ \hat{v}_{-k}^* & \hat{u}_{-k}^* \end{pmatrix} \tag{5}$$

$$= \frac{1}{2}\begin{pmatrix} \xi_k \hat{\sigma}_0 \hat{u}_k + \hat{\Delta}_k \hat{v}_{-k}^* & \xi_k \hat{\sigma}_0 \hat{v}_k + \hat{\Delta}_k \hat{u}_{-k}^* \\ \hat{\Delta}_k^\dagger \hat{u}_k - \xi_k \hat{\sigma}_0 \hat{v}_{-k}^* & \hat{\Delta}_k^\dagger \hat{v}_k - \xi_k \hat{\sigma}_0 \hat{u}_{-k}^* \end{pmatrix} \tag{6}$$

$$= \frac{1}{2}\begin{pmatrix} \xi_k \hat{u}_k + \hat{\Delta}_k \hat{v}_{-k}^* & \xi_k \hat{v}_k + \hat{\Delta}_k \hat{u}_{-k}^* \\ \hat{\Delta}_k^\dagger \hat{u}_k - \xi_k \hat{v}_{-k}^* & \hat{\Delta}_k^\dagger \hat{v}_k - \xi_k \hat{u}_{-k}^* \end{pmatrix}. \tag{7}$$

The right hand side is

$$\check{U}_k \check{E}_k = \begin{pmatrix} \hat{u}_k & \hat{v}_k \\ \hat{v}_{-k}^* & \hat{u}_{-k}^* \end{pmatrix} \frac{1}{2}\begin{pmatrix} \hat{E}_k & 0 \\ 0 & -\hat{E}_{-k} \end{pmatrix} \tag{8}$$

$$= \frac{1}{2}\begin{pmatrix} \hat{u}_k \hat{E}_k & -\hat{v}_k \hat{E}_{-k} \\ \hat{v}_{-k}^* \hat{E}_k & -\hat{u}_{-k}^* \hat{E}_{-k} \end{pmatrix}. \tag{9}$$

Therefore, from $\check{\varepsilon}_k \check{U}_k = \check{U}_k \check{E}_k$,

$$\frac{1}{2}\begin{pmatrix} \xi_k \hat{u}_k + \hat{\Delta}_k \hat{v}_{-k}^* & \xi_k \hat{v}_k + \hat{\Delta}_k \hat{u}_{-k}^* \\ \hat{\Delta}_k^\dagger \hat{u}_k - \xi_k \hat{v}_{-k}^* & \hat{\Delta}_k^\dagger \hat{v}_k - \xi_k \hat{u}_{-k}^* \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \hat{u}_k \hat{E}_k & -\hat{v}_k \hat{E}_{-k} \\ \hat{v}_{-k}^* \hat{E}_k & -\hat{u}_{-k}^* \hat{E}_{-k} \end{pmatrix}. \tag{10}$$

We obtain the four equations,

$$\xi_k \hat{u}_k + \hat{\Delta}_k \hat{v}_{-k}^* = \hat{u}_k \hat{E}_k, \tag{11}$$

$$\xi_k \hat{v}_k + \hat{\Delta}_k \hat{u}_{-k}^* = -\hat{v}_k \hat{E}_{-k}, \tag{12}$$

$$\hat{\Delta}_k^\dagger \hat{u}_k - \xi_k \hat{v}_{-k}^* = \hat{v}_{-k}^* \hat{E}_k, \tag{13}$$

$$\hat{\Delta}_k^\dagger \hat{v}_k - \xi_k \hat{u}_{-k}^* = -\hat{u}_{-k}^* \hat{E}_{-k}. \tag{14}$$

From Eq. (13),

$$\begin{aligned}
& \hat{\Delta}_k^\dagger \hat{u}_k - \xi_k \hat{v}_{-k}^* &=& \ \hat{v}_{-k}^* \hat{E}_k, && (15)\\
\rightarrow \quad & \xi_k \hat{v}_{-k}^* - \hat{\Delta}_k^\dagger \hat{u}_k &=& -\hat{v}_{-k}^* \hat{E}_k, && (16)\\
\rightarrow \quad & \xi_{-k} \hat{v}_k^* - \hat{\Delta}_{-k}^\dagger \hat{u}_{-k} &=& -\hat{v}_k^* \hat{E}_{-k}, \qquad (k \rightarrow -k) && (17)\\
\rightarrow \quad & \xi_k \hat{v}_k^* + \left(\hat{\Delta}_k^T\right)^\dagger \hat{u}_{-k} &=& -\hat{v}_k^* \hat{E}_{-k}, \qquad (\xi_{-k} = \xi_k \text{ and } \hat{\Delta}_{-k} = -\hat{\Delta}_k^T) && (18)\\
\rightarrow \quad & \xi_k \hat{v}_k^* + \hat{\Delta}_k^* \hat{u}_{-k} &=& -\hat{v}_k^* \hat{E}_{-k}, && (19)\\
\rightarrow \quad & \xi_k \hat{v}_k + \hat{\Delta}_k \hat{u}_{-k}^* &=& -\hat{v}_k \hat{E}_{-k}. && (20)
\end{aligned}$$

This is equivalent to Eq. (12), namely Eq. (13) is equivalent to Eq. (12).

From Eq. (14),

$$\begin{aligned}
& \hat{\Delta}_k^\dagger \hat{v}_k - \xi_k \hat{u}_{-k}^* &=& -\hat{u}_{-k}^* \hat{E}_{-k}, && (21)\\
\rightarrow \quad & \xi_k \hat{u}_{-k}^* - \hat{\Delta}_k^\dagger \hat{v}_k &=& \ \hat{u}_{-k}^* \hat{E}_{-k}, && (22)\\
\rightarrow \quad & \xi_{-k} \hat{u}_k^* - \hat{\Delta}_{-k}^\dagger \hat{v}_{-k} &=& \ \hat{u}_k^* \hat{E}_k, \qquad (k \rightarrow -k) && (23)\\
\rightarrow \quad & \xi_k \hat{u}_k^* + \left(\hat{\Delta}_k^T\right)^\dagger \hat{v}_{-k} &=& \ \hat{u}_k^* \hat{E}_k, \qquad (\xi_{-k} = \xi_k \text{ and } \hat{\Delta}_{-k} = -\hat{\Delta}_k^T) && (24)\\
\rightarrow \quad & \xi_k \hat{u}_k^* + \hat{\Delta}_k^* \hat{v}_{-k} &=& \ \hat{u}_k^* \hat{E}_k, && (25)\\
\rightarrow \quad & \xi_k \hat{u}_k + \hat{\Delta}_k \hat{v}_{-k}^* &=& \ \hat{u}_k \hat{E}_k. && (26)
\end{aligned}$$

This is equivalent to Eq. (11), namely Eq. (14) is equivalent to Eq. (11).

Thus, we obtain the two independent equations [Eq. (11) and (13)],

$$\begin{aligned}
\xi_k \hat{u}_k + \hat{\Delta}_k \hat{v}_{-k}^* &= \hat{u}_k \hat{E}_k, && (27)\\
\hat{\Delta}_k^\dagger \hat{u}_k - \xi_k \hat{v}_{-k}^* &= \hat{v}_{-k}^* \hat{E}_k. && (28)
\end{aligned}$$

Multiplying Eq. (28) by $\hat{\Delta}_k$ from the left gives

$$\hat{\Delta}_k \hat{\Delta}_k^\dagger \hat{u}_k - \xi_k \hat{\Delta}_k \hat{v}_{-k}^* = \hat{\Delta}_k \hat{v}_{-k}^* \hat{E}_k. \qquad (29)$$

Assuming the unitary states $\hat{\Delta}_k \hat{\Delta}_k^\dagger = \hat{\Delta}_k^\dagger \hat{\Delta}_k = |\Delta_k|^2 \hat{\sigma}_0$,

$$\begin{aligned}
& |\Delta_k|^2 \hat{u}_k - \xi_k \hat{\Delta}_k \hat{v}_{-k}^* &=& \ \hat{\Delta}_k \hat{v}_{-k}^* \hat{E}_k. && (30)\\
\rightarrow \quad & \hat{u}_k &=& \ \frac{1}{|\Delta_k|^2}\Big(\xi_k \hat{\Delta}_k \hat{v}_{-k}^* + \hat{\Delta}_k \hat{v}_{-k}^* \hat{E}_k\Big), && (31)
\end{aligned}$$

where $|\Delta_k|^2 \equiv \frac{1}{2}\mathrm{Tr}[\hat{\Delta}_k \hat{\Delta}_k^\dagger]$. Substituting this $\hat{u}_k$ into Eq. (27),

$$\frac{\xi_k}{|\Delta_k|^2}\Big(\xi_k \hat{\Delta}_k \hat{v}_{-k}^* + \hat{\Delta}_k \hat{v}_{-k}^* \hat{E}_k\Big) + \hat{\Delta}_k \hat{v}_{-k}^* = \frac{1}{|\Delta_k|^2}\Big(\xi_k \hat{\Delta}_k \hat{v}_{-k}^* + \hat{\Delta}_k \hat{v}_{-k}^* \hat{E}_k\Big)\hat{E}_k, \qquad (32)$$

$$\begin{aligned}
\rightarrow \quad & \frac{\xi_k}{|\Delta_k|^2}\Big(\xi_k (\hat{\Delta}_k^\dagger \hat{\Delta}_k) \hat{v}_{-k}^* + (\hat{\Delta}_k^\dagger \hat{\Delta}_k) \hat{v}_{-k}^* \hat{E}_k\Big) + (\hat{\Delta}_k^\dagger \hat{\Delta}_k) \hat{v}_{-k}^* && (33)\\
& = \frac{1}{|\Delta_k|^2}\Big(\xi_k (\hat{\Delta}_k^\dagger \hat{\Delta}_k) \hat{v}_{-k}^* + (\hat{\Delta}_k^\dagger \hat{\Delta}_k) \hat{v}_{-k}^* \hat{E}_k\Big)\hat{E}_k, && (34)
\end{aligned}$$

$$\rightarrow \quad \xi_k\Big(\xi_k\hat{v}^*_{-k}+\hat{v}^*_{-k}\hat{E}_k\Big)+|\Delta_k|^2\hat{v}^*_{-k} = \Big(\xi_k\hat{v}^*_{-k}+\hat{v}^*_{-k}\hat{E}_k\Big)\hat{E}_k, \tag{35}$$

$$\rightarrow \quad \hat{v}^*_{-k}\Big(\xi_k^2\hat{\sigma}_0+\xi_k\hat{E}_k+|\Delta_k|^2\hat{\sigma}_0\Big) = \hat{v}^*_{-k}\Big(\xi_k\hat{E}_k+\hat{E}_k^2\Big), \tag{36}$$

For $\hat{v}^*_{-k} \neq 0$,

$$\xi_k^2\hat{\sigma}_0+\xi_k\hat{E}_k+|\Delta_k|^2\hat{\sigma}_0 = \xi_k\hat{E}_k+\hat{E}_k^2, \tag{37}$$

$$\rightarrow \quad (\xi_k^2+|\Delta_k|^2)\hat{\sigma}_0, = \hat{E}_k^2 \tag{38}$$

$$\rightarrow \quad (\xi_k^2+|\Delta_k|^2)\hat{\sigma}_0, = \begin{pmatrix} E_{k,+} & 0 \\ 0 & E_{k,-} \end{pmatrix}^2, \tag{39}$$

$$\rightarrow \quad (\xi_k^2+|\Delta_k|^2)\hat{\sigma}_0, = \begin{pmatrix} E_{k,+}^2 & 0 \\ 0 & E_{k,-}^2 \end{pmatrix}. \tag{40}$$

Therefore,

$$E_{k,+}^2 = E_{k,-}^2 = \xi_k^2+|\Delta_k|^2. \tag{41}$$

Now, we define

$$E_k \equiv \sqrt{\xi_k^2+|\Delta_k|^2}. \tag{42}$$

Then,

$$E_{-k} = \sqrt{\xi_{-k}^2+|\Delta_{-k}|^2} \tag{43}$$

$$= \sqrt{\xi_k^2+|\Delta_k|^2} \tag{44}$$

$$= E_k. \tag{45}$$

Let us consider

$$\hat{E}_k = \begin{pmatrix} E_{k,+} & 0 \\ 0 & E_{k,-} \end{pmatrix} = \begin{pmatrix} E_k & 0 \\ 0 & E_k \end{pmatrix} = E_k\hat{\sigma}_0. \tag{46}$$

Substituting this into Eqs. (27) and (28),

$$\xi_k\hat{u}_k+\hat{\Delta}_k\hat{v}^*_{-k} = E_k\hat{u}_k, \tag{47}$$

$$\hat{\Delta}_k^\dagger\hat{u}_k-\xi_k\hat{v}^*_{-k} = E_k\hat{v}^*_{-k}. \tag{48}$$

$$\rightarrow \quad \hat{\Delta}_k\hat{v}^*_{-k} = (E_k-\xi_k)\hat{u}_k, \tag{49}$$

$$\hat{\Delta}_k^\dagger\hat{u}_k = (E_k+\xi_k)\hat{v}^*_{-k}. \tag{50}$$

On the other hand, because $\check{U}_k$ is a unitary matrix,

$$\check{1} = \check{U}_k\check{U}_k^\dagger \tag{51}$$

$$= \begin{pmatrix} \hat{u}_k & \hat{v}_k \\ \hat{v}^*_{-k} & \hat{u}^*_{-k} \end{pmatrix}\begin{pmatrix} \hat{u}_k^\dagger & \hat{v}^{*\dagger}_{-k} \\ \hat{v}_k^\dagger & \hat{u}^{*\dagger}_{-k} \end{pmatrix} \tag{52}$$

$$= \begin{pmatrix} \hat{u}_k & \hat{v}_k \\ \hat{v}^*_{-k} & \hat{u}^*_{-k} \end{pmatrix}\begin{pmatrix} \hat{u}_k^\dagger & \hat{v}^{T}_{-k} \\ \hat{v}_k^\dagger & \hat{u}^{T}_{-k} \end{pmatrix} \tag{53}$$

$$= \begin{pmatrix} \hat{u}_k\hat{u}_k^\dagger+\hat{v}_k\hat{v}_k^\dagger & \hat{u}_k\hat{v}^T_{-k}+\hat{v}_k\hat{u}^T_{-k} \\ \hat{v}^*_{-k}\hat{u}_k^\dagger+\hat{u}^*_{-k}\hat{v}_k^\dagger & \hat{v}^*_{-k}\hat{v}^T_{-k}+\hat{u}^*_{-k}\hat{u}^T_{-k} \end{pmatrix}. \tag{54}$$

Thus,

$$\hat{u}_k\hat{u}_k^\dagger + \hat{v}_k\hat{v}_k^\dagger = \hat{\sigma}_0, \tag{55}$$

$$\hat{v}_{-k}^*\hat{u}_k^\dagger + \hat{u}_{-k}^*\hat{v}_k^\dagger = 0. \tag{56}$$

From Eq. (49),

$$\hat{u}_k = \frac{\hat{\Delta}_k}{E_k - \xi_k}\hat{v}_{-k}^*, \tag{57}$$

$$\hat{u}_{-k}^* = \frac{\hat{\Delta}_{-k}^*}{E_{-k} - \xi_{-k}}\hat{v}_k \tag{58}$$

$$= \frac{-\hat{\Delta}_k^\dagger}{E_k - \xi_k}\hat{v}_k. \tag{59}$$

From Eq. (50),

$$\hat{v}_{-k}^* = \frac{\hat{\Delta}_k^\dagger}{E_k + \xi_k}\hat{u}_k, \tag{60}$$

$$\hat{v}_k = \frac{\hat{\Delta}_{-k}^T}{E_{-k} + \xi_{-k}}\hat{u}_{-k}^* \tag{61}$$

$$= \frac{-\hat{\Delta}_k}{E_k + \xi_k}\hat{u}_{-k}^*. \tag{62}$$

Substituting these into Eq. (56),

$$\Big(\frac{\hat{\Delta}_k^\dagger}{E_k + \xi_k}\hat{u}_k\Big)\hat{u}_k^\dagger + \Big(\frac{-\hat{\Delta}_k^\dagger}{E_k - \xi_k}\hat{v}_k\Big)\hat{v}_k^\dagger = 0, \tag{63}$$

$$\rightarrow \quad (E_k - \xi_k)\hat{\Delta}_k\hat{\Delta}_k^\dagger\hat{u}_k\hat{u}_k^\dagger - (E_k + \xi_k)\hat{\Delta}_k\hat{\Delta}_k^\dagger\hat{v}_k\hat{v}_k^\dagger = 0, \tag{64}$$

$$\rightarrow \quad (E_k - \xi_k)|\Delta_k|^2\hat{u}_k\hat{u}_k^\dagger - (E_k + \xi_k)|\Delta_k|^2\hat{v}_k\hat{v}_k^\dagger = 0, \tag{65}$$

$$\rightarrow \quad (E_k - \xi_k)\hat{u}_k\hat{u}_k^\dagger - (E_k + \xi_k)\hat{v}_k\hat{v}_k^\dagger = 0, \tag{66}$$

$$\rightarrow \quad E_k(\hat{u}_k\hat{u}_k^\dagger - \hat{v}_k\hat{v}_k^\dagger) - \xi_k(\hat{u}_k\hat{u}_k^\dagger + \hat{v}_k\hat{v}_k^\dagger) = 0. \tag{67}$$

Substituting Eq. (55) into this,

$$E_k\Big(\hat{u}_k\hat{u}_k^\dagger - (\hat{\sigma}_0 - \hat{u}_k\hat{u}_k^\dagger)\Big) - \xi_k\hat{\sigma}_0 = 0 \tag{68}$$

$$\rightarrow \quad 2E_k\hat{u}_k\hat{u}_k^\dagger = (E_k + \xi_k)\hat{\sigma}_0, \tag{69}$$

or

$$E_k\Big((\hat{\sigma}_0 - \hat{v}_k\hat{v}_k^\dagger) - \hat{v}_k\hat{v}_k^\dagger\Big) - \xi_k\hat{\sigma}_0 = 0 \tag{70}$$

$$\rightarrow \quad 2E_k\hat{v}_k\hat{v}_k^\dagger = (E_k - \xi_k)\hat{\sigma}_0. \tag{71}$$

Provided that $\underline{\hat{u}_k = u_k\hat{\sigma}_0 \text{ and } u_k \text{ is real}}$, then from Eq. (69),

$$\hat{u}_k = \sqrt{\frac{1}{2}\Big(1 + \frac{\xi_k}{E_k}\Big)}\hat{\sigma}_0 \tag{72}$$

$$= \sqrt{\frac{E_k + \xi_k}{2E_k}}\hat{\sigma}_0 \tag{73}$$

$$= \frac{(E_k + \xi_k)\hat{\sigma}_0}{\sqrt{2E_k(E_k + \xi_k)}}. \tag{74}$$

Substituting this into Eq. (62),

$$\hat{v}_k = \frac{-\hat{\Delta}_k}{E_k+\xi_k}\hat{u}^*_{-k} \tag{75}$$

$$= \frac{-\hat{\Delta}_k}{E_k+\xi_k}\sqrt{\frac{1}{2}\Big(1+\frac{\xi_{-k}}{E_{-k}}\Big)}\hat{\sigma}_0 \tag{76}$$

$$= \frac{-\hat{\Delta}_k}{E_k+\xi_k}\sqrt{\frac{1}{2}\Big(1+\frac{\xi_k}{E_k}\Big)}\hat{\sigma}_0 \tag{77}$$

$$= \frac{-\hat{\Delta}_k}{E_k+\xi_k}\sqrt{\frac{E_k+\xi_k}{2E_k}}\hat{\sigma}_0 \tag{78}$$

$$= \frac{-\hat{\Delta}_k}{\sqrt{2E_k(E_k+\xi_k)}}. \tag{79}$$

Appendix I:

In the case of singlet states,

$$\hat{\Delta}_k = \Delta_k i\hat{\sigma}^y \tag{80}$$

$$= \sqrt{E_k^2-\xi_k^2}\, i\hat{\sigma}^y. \tag{81}$$

Here, we have used $E_k \equiv \sqrt{\xi_k^2+|\Delta_k|^2}$ [Eq. (42)], and have assumed Δ_k is real for simplicity.

Substituting this into Eq. (79),

$$\hat{v}_k = \frac{-\hat{\Delta}_k}{\sqrt{2E_k(E_k+\xi_k)}} \tag{82}$$

$$= \frac{-\sqrt{E_k^2-\xi_k^2}\, i\hat{\sigma}^y}{\sqrt{2E_k(E_k+\xi_k)}} \tag{83}$$

$$= \sqrt{\frac{E_k^2-\xi_k^2}{2E_k(E_k+\xi_k)}}(-i\hat{\sigma}^y) \tag{84}$$

$$= \sqrt{\frac{(E_k-\xi_k)(E_k+\xi_k)}{2E_k(E_k+\xi_k)}}(-i\hat{\sigma}^y) \tag{85}$$

$$= \sqrt{\frac{(E_k-\xi_k)}{2E_k}}(-i\hat{\sigma}^y) \tag{86}$$

$$= \sqrt{\frac{1}{2}\Big(1-\frac{\xi_k}{E_k}\Big)}(-i\hat{\sigma}^y). \tag{87}$$

Appendix II:

Provided that $\underline{\hat{v}_k = v_k\hat{\sigma}_0 \text{ and } v_k \text{ is real}}$, then from Eq. (71),

$$\hat{v}_k = \sqrt{\frac{1}{2}\Big(1 - \frac{\xi_k}{E_k}\Big)}\hat{\sigma}_0 \tag{88}$$

$$= \sqrt{\frac{E_k - \xi_k}{2E_k}}\hat{\sigma}_0 \tag{89}$$

$$= \frac{(E_k - \xi_k)\hat{\sigma}_0}{\sqrt{2E_k(E_k - \xi_k)}}. \tag{90}$$

Substituting this into Eq. (57),

$$\hat{u}_k = \frac{\hat{\Delta}_k}{E_k - \xi_k}\hat{v}^*_{-k} \tag{91}$$

$$= \frac{\hat{\Delta}_k}{E_k - \xi_k}\frac{(E_{-k} - \xi_{-k})\hat{\sigma}_0}{\sqrt{2E_{-k}(E_{-k} - \xi_{-k})}} \tag{92}$$

$$= \frac{\hat{\Delta}_k}{E_k - \xi_k}\frac{(E_k - \xi_k)\hat{\sigma}_0}{\sqrt{2E_k(E_k - \xi_k)}} \tag{93}$$

$$= \frac{\hat{\Delta}_k}{\sqrt{2E_k(E_k - \xi_k)}}. \tag{94}$$

In the case of singlet states, assuming Δ_k is real for simplicity,

$$\hat{u}_k = \frac{\hat{\Delta}_k}{\sqrt{2E_k(E_k - \xi_k)}} \tag{95}$$

$$= \frac{\sqrt{E_k^2 - \xi_k^2}i\hat{\sigma}^y}{\sqrt{2E_k(E_k - \xi_k)}} \tag{96}$$

$$= \sqrt{\frac{E_k^2 - \xi_k^2}{2E_k(E_k - \xi_k)}}(i\hat{\sigma}^y) \tag{97}$$

$$= \sqrt{\frac{(E_k - \xi_k)(E_k + \xi_k)}{2E_k(E_k - \xi_k)}}(i\hat{\sigma}^y) \tag{98}$$

$$= \sqrt{\frac{(E_k + \xi_k)}{2E_k}}(i\hat{\sigma}^y) \tag{99}$$

$$= \sqrt{\frac{1}{2}\Big(1 + \frac{\xi_k}{E_k}\Big)}(i\hat{\sigma}^y). \tag{100}$$

4.2 Consider the gap equation

$$\Delta_{k,s_1s_2} = -\sum_{k',s_3s_4} V_{k,k';s_1s_2s_3s_4} \frac{\Delta_{k',s_4s_3}}{2E_{k'}} \tanh\Big(\frac{E_{k'}}{2k_{\rm B}T}\Big), \tag{101}$$

and the pairing interaction of the form

$$V_{k,k';s_1s_2s_3s_4} = J^0_{k,k'}\sigma^0_{s_1s_4}\sigma^0_{s_2s_3} + J_{k,k'}\vec{\sigma}_{s_1s_4}\cdot\vec{\sigma}_{s_2s_3}. \tag{102}$$

Here, $\hat{\sigma}^0$ is the 2×2 unit matrix and $\hat{\vec{\sigma}} = (\hat{\sigma}^x, \hat{\sigma}^y, \hat{\sigma}^z)$ are the Pauli matrices. $A_{ss'}$ denotes the ss' matrix element of $\hat{A}$, $(s, s' = \{\uparrow, \downarrow\})$. $\hat{A}$ stands for $\hat{\Delta}_k$, $\hat{\sigma}^0$, and $\hat{\vec{\sigma}}$.

Then,

$$\vec{\sigma}_{s_1s_4}\cdot\vec{\sigma}_{s_2s_3} = \sigma^x_{s_1s_4}\sigma^x_{s_2s_3} + \sigma^y_{s_1s_4}\sigma^y_{s_2s_3} + \sigma^z_{s_1s_4}\sigma^z_{s_2s_3}. \tag{103}$$

$$\begin{aligned}
\sum_{s_3s_4} V_{k,k';s_1s_2s_3s_4}\Delta_{k',s_4s_3} &= \sum_{s_3s_4}[J^0_{k,k'}\sigma^0_{s_1s_4}\sigma^0_{s_2s_3} + J_{k,k'}\hat{\vec{\sigma}}_{s_1s_4}\cdot\hat{\vec{\sigma}}_{s_2s_3}]\Delta_{k',s_4s_3} &(104)\\
&= \sum_{s_3s_4}\Big[J^0_{k,k'}\sigma^0_{s_1s_4}\sigma^0_{s_2s_3} \\
&\quad + J_{k,k'}\{\sigma^x_{s_1s_4}\sigma^x_{s_2s_3} + \sigma^y_{s_1s_4}\sigma^y_{s_2s_3} + \sigma^z_{s_1s_4}\sigma^z_{s_2s_3}\}\Big]\Delta_{k',s_4s_3} &(105)\\
&= \sum_{s_3s_4}\Big[J^0_{k,k'}\sigma^0_{s_1s_4}\Delta_{k',s_4s_3}\sigma^0_{s_2s_3} \\
&\quad + J_{k,k'}\{\sigma^x_{s_1s_4}\Delta_{k',s_4s_3}\sigma^x_{s_2s_3} + \sigma^y_{s_1s_4}\Delta_{k',s_4s_3}\sigma^y_{s_2s_3} + \sigma^z_{s_1s_4}\Delta_{k',s_4s_3}\sigma^z_{s_2s_3}\}\Big] \\
& &(106)\\
&= \sum_{s_3s_4}\Big[J^0_{k,k'}\sigma^0_{s_1s_4}\Delta_{k',s_4s_3}\sigma^0_{s_3s_2} \\
&\quad + J_{k,k'}\{\sigma^x_{s_1s_4}\Delta_{k',s_4s_3}\sigma^x_{s_3s_2} + \sigma^y_{s_1s_4}\Delta_{k',s_4s_3}(-\sigma^y_{s_3s_2}) + \sigma^z_{s_1s_4}\Delta_{k',s_4s_3}\sigma^z_{s_3s_2}\}\Big] \\
& &(107)\\
&= \Big[J^0_{k,k'}[\hat{\sigma}^0\hat{\Delta}_{k'}\hat{\sigma}^0]_{s_1s_2} \\
&\quad + J_{k,k'}\Big\{[\hat{\sigma}^x\hat{\Delta}_{k'}\hat{\sigma}^x]_{s_1s_2} - [\hat{\sigma}^y\hat{\Delta}_{k'}\hat{\sigma}^y]_{s_1s_2} + [\hat{\sigma}^z\hat{\Delta}_{k'}\hat{\sigma}^z]_{s_1s_2}\Big\}\Big]. &(108)
\end{aligned}$$

For singlet states, the order parameter is written as

$$\begin{aligned}
\hat{\Delta}_k &= \Psi_k i\hat{\sigma}^y &(109)\\
&= \begin{pmatrix} 0 & \Psi_k \\ -\Psi_k & 0 \end{pmatrix}. &(110)
\end{aligned}$$

In this case, from Eq. (108),

$$\begin{aligned}
\sum_{s_3s_4} V_{k,k';s_1s_2s_3s_4}\Delta_{k',s_4s_3} &= \Big[J^0_{k,k'}[\hat{\sigma}^0(i\hat{\sigma}^y)\hat{\sigma}^0]_{s_1s_2} \\
&\quad + J_{k,k'}\Big\{[\hat{\sigma}^x(i\hat{\sigma}^y)\hat{\sigma}^x]_{s_1s_2} - [\hat{\sigma}^y(i\hat{\sigma}^y)\hat{\sigma}^y]_{s_1s_2} + [\hat{\sigma}^z(i\hat{\sigma}^y)\hat{\sigma}^z]_{s_1s_2}\Big\}\Big]\Psi_{k'} \\
& &(111)\\
&= \Big[J^0_{k,k'}i\hat{\sigma}^y_{s_1s_2} \\
&\quad + J_{k,k'}\Big\{i[\hat{\sigma}^x(-i\hat{\sigma}^z)]_{s_1s_2} - i[\hat{\sigma}^y\hat{\sigma}^0]_{s_1s_2} + i[\hat{\sigma}^z(i\hat{\sigma}^x)]_{s_1s_2}\Big\}\Big]\Psi_{k'} &(112)
\end{aligned}$$

$$
\begin{aligned}
&= \Big[J^0_{k,k'} i\hat{\sigma}^y_{s_1 s_2} \\
&\quad + J_{k,k'}\Big\{i(-i)[-i\hat{\sigma}^y]_{s_1 s_2} - i\sigma^y_{s_1 s_2} + i^2[i\hat{\sigma}^y]_{s_1 s_2}\Big\}\Big]\Psi_{k'} \qquad (113)\\
&= \Big[J^0_{k,k'} i\sigma^y_{s_1 s_2} \\
&\quad + J_{k,k'}\Big\{-i\sigma^y_{s_1 s_2} - i\sigma^y_{s_1 s_2} - i\sigma^y_{s_1 s_2}\Big\}\Big]\Psi_{k'} \qquad (114)\\
&= (J^0_{k,k'} - 3J_{k,k'})\Psi_{k'} i\sigma^y_{s_1 s_2}. \qquad (115)
\end{aligned}
$$

Therefore, substituting this and $\hat{\Delta}_k = \Psi_k i\hat{\sigma}^y$ into Eq. (101),

$$
\begin{aligned}
\Psi_k i\sigma^y_{s_1 s_2} &= -\sum_{k'}(J^0_{k,k'} - 3J_{k,k'})\frac{\Psi_{k'} i\sigma^y_{s_1 s_2}}{2E_{k'}}\tanh\Big(\frac{E_{k'}}{2k_{\rm B}T}\Big), \qquad (116)\\
\rightarrow \qquad \Psi_k &= -\sum_{k'}(J^0_{k,k'} - 3J_{k,k'})\frac{\Psi_{k'}}{2E_{k'}}\tanh\Big(\frac{E_{k'}}{2k_{\rm B}T}\Big). \qquad (117)
\end{aligned}
$$

Next, let us consider triplet states. The order parameter is written as

$$
\begin{aligned}
\hat{\Delta}_k &= \vec{d}_k \cdot \hat{\vec{\sigma}} i\sigma^y \qquad (118)\\
&= i d^x_k \hat{\sigma}^x\hat{\sigma}^y + i d^y_k \hat{\sigma}^y\hat{\sigma}^y + i d^z_k \hat{\sigma}^z\hat{\sigma}^y \qquad (119)\\
&= i d^x_k (i\hat{\sigma}^z) + i d^y_k \hat{\sigma}^0 + i d^z_k(-i\hat{\sigma}^x) \qquad (120)\\
&= -d^x_k\hat{\sigma}^z + i d^y_k\hat{\sigma}^0 + d^z_k\hat{\sigma}^x \qquad (121)\\
&= \begin{pmatrix} -d^x_k + i d^y_k & d^z_k \\ d^z_k & d^x_k + i d^y_k \end{pmatrix}. \qquad (122)
\end{aligned}
$$

In this case, from Eq. (108),

$$
\begin{aligned}
\sum_{s_3 s_4} V_{k,k';s_1 s_2 s_3 s_4}\Delta_{k',s_4 s_3} &= J^0_{k,k'}[\hat{\sigma}^0(-d^x_{k'}\hat{\sigma}^z + i d^y_{k'}\hat{\sigma}^0 + d^z_{k'}\hat{\sigma}^x)\hat{\sigma}^0]_{s_1 s_2} \\
&\quad + J_{k,k'}\Big\{[\hat{\sigma}^x(-d^x_{k'}\hat{\sigma}^z + i d^y_{k'}\hat{\sigma}^0 + d^z_{k'}\hat{\sigma}^x)\hat{\sigma}^x]_{s_1 s_2} \\
&\qquad - [\hat{\sigma}^y(-d^x_{k'}\hat{\sigma}^z + i d^y_{k'}\hat{\sigma}^0 + d^z_{k'}\hat{\sigma}^x)\hat{\sigma}^y]_{s_1 s_2} \\
&\qquad + [\hat{\sigma}^z(-d^x_{k'}\hat{\sigma}^z + i d^y_{k'}\hat{\sigma}^0 + d^z_{k'}\hat{\sigma}^x)\hat{\sigma}^z]_{s_1 s_2}\Big\} \qquad (123)\\
&= J^0_{k,k'}[(-d^x_{k'}\hat{\sigma}^z + i d^y_{k'}\hat{\sigma}^0 + d^z_{k'}\hat{\sigma}^x)]_{s_1 s_2} \\
&\quad + J_{k,k'}\Big\{[\hat{\sigma}^x(-d^x_{k'} i\hat{\sigma}^y + i d^y_{k'}\hat{\sigma}^x + d^z_{k'}\hat{\sigma}^0)]_{s_1 s_2} \\
&\qquad - [\hat{\sigma}^y(-d^x_{k'}(-i)\hat{\sigma}^x + i d^y_{k'}\hat{\sigma}^y + d^z_{k'} i\hat{\sigma}^z)]_{s_1 s_2} \\
&\qquad + [\hat{\sigma}^z(-d^x_{k'}\hat{\sigma}^0 + i d^y_{k'}\hat{\sigma}^z + d^z_{k'}(-i)\hat{\sigma}^y)]_{s_1 s_2}\Big\} \qquad (124)\\
&= J^0_{k,k'}(-d^x_{k'}\sigma^z_{s_1 s_2} + i d^y_{k'}\sigma^0_{s_1 s_2} + d^z_{k'}\sigma^x_{s_1 s_2}) \\
&\quad + J_{k,k'}\Big\{[(-d^x_{k'} i^2\hat{\sigma}^z + i d^y_{k'}\hat{\sigma}^0 + d^z_{k'}\hat{\sigma}^x)]_{s_1 s_2} \\
&\qquad - [(-d^x_{k'}(-i)^2\hat{\sigma}^z + i d^y_{k'}\hat{\sigma}^0 + d^z_{k'} i^2\hat{\sigma}^x)]_{s_1 s_2} \\
&\qquad + [(-d^x_{k'}\hat{\sigma}^z + i d^y_{k'}\hat{\sigma}^0 + d^z_{k'}(-i)^2\hat{\sigma}^x)]_{s_1 s_2} \qquad (125)\\
&= J^0_{k,k'}(-d^x_{k'}\sigma^z_{s_1 s_2} + i d^y_{k'}\sigma^0_{s_1 s_2} + d^z_{k'}\sigma^x_{s_1 s_2}) \\
&\quad + J_{k,k'}\Big\{(d^x_{k'}\sigma^z_{s_1 s_2} + i d^y_{k'}\sigma^0_{s_1 s_2} + d^z_{k'}\sigma^x_{s_1 s_2}) \\
&\qquad - (d^x_{k'}\sigma^z_{s_1 s_2} + i d^y_{k'}\sigma^0_{s_1 s_2} - d^z_{k'}\sigma^x_{s_1 s_2}) \\
&\qquad + (-d^x_{k'}\sigma^z_{s_1 s_2} + i d^y_{k'}\sigma^0_{s_1 s_2} - d^z_{k'}\sigma^x_{s_1 s_2})\Big\} \qquad (126)
\end{aligned}
$$

$$
\begin{aligned}
&= J^0_{k,k'}(-d^x_{k'}\sigma^z_{s_1s_2} + id^y_{k'}\sigma^0_{s_1s_2} + d^z_{k'}\sigma^x_{s_1s_2}) \\
&\quad + J_{k,k'}\Big\{(d^x_{k'}\sigma^z_{s_1s_2} + id^y_{k'}\sigma^0_{s_1s_2} + d^z_{k'}\sigma^x_{s_1s_2}) \\
&\qquad + (-d^x_{k'}\sigma^z_{s_1s_2} - id^y_{k'}\sigma^0_{s_1s_2} + d^z_{k'}\sigma^x_{s_1s_2}) \\
&\qquad + (-d^x_{k'}\sigma^z_{s_1s_2} + id^y_{k'}\sigma^0_{s_1s_2} - d^z_{k'}\sigma^x_{s_1s_2})\Big\} \qquad (127) \\
&= J^0_{k,k'}(-d^x_{k'}\sigma^z_{s_1s_2} + id^y_{k'}\sigma^0_{s_1s_2} + d^z_{k'}\sigma^x_{s_1s_2}) \\
&\quad + J_{k,k'}(-d^x_{k'}\sigma^z_{s_1s_2} + id^y_{k'}\sigma^0_{s_1s_2} + d^z_{k'}\sigma^x_{s_1s_2}) \qquad (128) \\
&= (J^0_{k,k'} + J_{k,k'})(-d^x_{k'}\sigma^z_{s_1s_2} + id^y_{k'}\sigma^0_{s_1s_2} + d^z_{k'}\sigma^x_{s_1s_2}) \qquad (129) \\
&= (J^0_{k,k'} + J_{k,k'})[-d^x_{k'}\hat{\sigma}^z + id^y_{k'}\hat{\sigma}^0 + d^z_{k'}\hat{\sigma}^x]_{s_1s_2} \qquad (130) \\
&= (J^0_{k,k'} + J_{k,k'})\vec{d}_{k'} \cdot \hat{\vec{\sigma}} i\sigma^y_{s_1s_2}. \qquad (131)
\end{aligned}
$$

Therefore, substituting this and $\hat{\Delta}_k = \vec{d}_k \cdot \hat{\vec{\sigma}} i\sigma^y$ into Eq. (101),

$$
\begin{aligned}
\vec{d}_k \cdot \hat{\vec{\sigma}} i\sigma^y_{s_1s_2} &= -\sum_{k'}(J^0_{k,k'} + J_{k,k'})\frac{\vec{d}_{k'} \cdot \hat{\vec{\sigma}} i\sigma^y_{s_1s_2}}{2E_{k'}}\tanh\Big(\frac{E_{k'}}{2k_{\mathrm{B}}T}\Big), \qquad (132) \\
\rightarrow \qquad \vec{d}_k &= -\sum_{k'}(J^0_{k,k'} + J_{k,k'})\frac{\vec{d}_{k'}}{2E_{k'}}\tanh\Big(\frac{E_{k'}}{2k_{\mathrm{B}}T}\Big). \qquad (133)
\end{aligned}
$$