Unkonventionelle Supraleitung WS 05/06
Lösungen zur Serie 5

5.1

$$\begin{aligned}
\mathrm{Tr}\Big[\hat{\Delta}_k^\dagger \hat{\sigma}_x \hat{\Delta}_k\Big] &= \mathrm{Tr}\Big[(\vec{d}_k\cdot\hat{\vec{\sigma}} i\hat{\sigma}_y)^\dagger \hat{\sigma}_x (\vec{d}_k\cdot\hat{\vec{\sigma}} i\hat{\sigma}_y)\Big] && (1)\\
&= \mathrm{Tr}\Big[(\vec{d}_k\cdot\hat{\vec{\sigma}} i\hat{\sigma}_y)^\dagger \hat{\sigma}_x (d_k^x\hat{\sigma}_x + d_k^y\hat{\sigma}_y + d_k^z\hat{\sigma}_z) i\hat{\sigma}_y\Big] && (2)\\
&= \mathrm{Tr}\Big[(\vec{d}_k\cdot\hat{\vec{\sigma}} i\hat{\sigma}_y)^\dagger \{d_k^x\hat{\sigma}_0 + d_k^y i\hat{\sigma}_z + d_k^z(-i\hat{\sigma}_y)\} i\hat{\sigma}_y\Big] && (3)\\
&= \mathrm{Tr}\Big[-i\hat{\sigma}_y (d_k^{x*}\hat{\sigma}_x + d_k^{y*}\hat{\sigma}_y + d_k^{z*}\hat{\sigma}_z)\{d_k^x\hat{\sigma}_0 + d_k^y i\hat{\sigma}_z + d_k^z(-i\hat{\sigma}_y)\} i\hat{\sigma}_y\Big] && (4)\\
&= \mathrm{Tr}\Big[-i\hat{\sigma}_y\{d_k^{x*}d_k^x\hat{\sigma}_x + d_k^{x*}d_k^y i\cdot(-i)\hat{\sigma}_y + d_k^{x*}d_k^z(-i)\cdot i\hat{\sigma}_z \\
&\qquad + d_k^{y*}d_k^x\hat{\sigma}_y + d_k^{y*}d_k^y i^2\hat{\sigma}_x + d_k^{y*}d_k^z(-i)\hat{\sigma}_0 \\
&\qquad + d_k^{z*}d_k^x\hat{\sigma}_z + d_k^{z*}d_k^y i\hat{\sigma}_0 + d_k^{z*}d_k^z(-i)^2\hat{\sigma}_x\} i\hat{\sigma}_y\Big] && (5)\\
&= \mathrm{Tr}\Big[-i\hat{\sigma}_y\{|d_k^x|^2\hat{\sigma}_x + d_k^{x*}d_k^y\hat{\sigma}_y + d_k^{x*}d_k^z\hat{\sigma}_z \\
&\qquad + d_k^{y*}d_k^x\hat{\sigma}_y - |d_k^y|^2\hat{\sigma}_x - i d_k^{y*}d_k^z\hat{\sigma}_0 \\
&\qquad + d_k^{z*}d_k^x\hat{\sigma}_z + i d_k^{z*}d_k^y\hat{\sigma}_0 - |d_k^z|^2\hat{\sigma}_x\} i\hat{\sigma}_y\Big] && (6)\\
&= \mathrm{Tr}\Big[-|d_k^x|^2\hat{\sigma}_x + d_k^{x*}d_k^y\hat{\sigma}_y - d_k^{x*}d_k^z\hat{\sigma}_z \\
&\qquad + d_k^{y*}d_k^x\hat{\sigma}_y + |d_k^y|^2\hat{\sigma}_x - i d_k^{y*}d_k^z\hat{\sigma}_0 \\
&\qquad - d_k^{z*}d_k^x\hat{\sigma}_z + i d_k^{z*}d_k^y\hat{\sigma}_0 + |d_k^z|^2\hat{\sigma}_x\Big] && (7)\\
&= \mathrm{Tr}\Big[-i d_k^{y*}d_k^z\hat{\sigma}_0 + i d_k^{z*}d_k^y\hat{\sigma}_0\Big] && (8)\\
&= -2i(d_k^{y*}d_k^z - d_k^{z*}d_k^y) && (9)\\
&= -2i[\vec{d}_k^*\times\vec{d}_k]_x. && (10)
\end{aligned}$$

Here, in Eq. (7) we have used the relations: $\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_y = -\hat{\sigma}_x$, $\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y = \hat{\sigma}_y$, $\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_y = -\hat{\sigma}_z$, and $\hat{\sigma}_y\hat{\sigma}_0\hat{\sigma}_y = \hat{\sigma}_0$.

$$\begin{aligned}
\mathrm{Tr}\Big[\hat{\Delta}_k^\dagger \hat{\sigma}_y \hat{\Delta}_k\Big] &= \mathrm{Tr}\Big[(\vec{d}_k\cdot\hat{\vec{\sigma}} i\hat{\sigma}_y)^\dagger \hat{\sigma}_y (\vec{d}_k\cdot\hat{\vec{\sigma}} i\hat{\sigma}_y)\Big] && (11)\\
&= \mathrm{Tr}\Big[(\vec{d}_k\cdot\hat{\vec{\sigma}} i\hat{\sigma}_y)^\dagger \hat{\sigma}_y (d_k^x\hat{\sigma}_x + d_k^y\hat{\sigma}_y + d_k^z\hat{\sigma}_z) i\hat{\sigma}_y\Big] && (12)\\
&= \mathrm{Tr}\Big[(\vec{d}_k\cdot\hat{\vec{\sigma}} i\hat{\sigma}_y)^\dagger \{d_k^x(-i\hat{\sigma}_z) + d_k^y\hat{\sigma}_0 + d_k^z i\hat{\sigma}_x\} i\hat{\sigma}_y\Big] && (13)\\
&= \mathrm{Tr}\Big[-i\hat{\sigma}_y (d_k^{x*}\hat{\sigma}_x + d_k^{y*}\hat{\sigma}_y + d_k^{z*}\hat{\sigma}_z)\{d_k^x(-i\hat{\sigma}_z) + d_k^y\hat{\sigma}_0 + d_k^z i\hat{\sigma}_x\} i\hat{\sigma}_y\Big] && (14)\\
&= \mathrm{Tr}\Big[-i\hat{\sigma}_y\{d_k^{x*}d_k^x(-i)^2\hat{\sigma}_y + d_k^{x*}d_k^y\hat{\sigma}_x + d_k^{x*}d_k^z i\hat{\sigma}_0 \\
&\qquad + d_k^{y*}d_k^x(-i)\cdot i\hat{\sigma}_x + d_k^{y*}d_k^y\hat{\sigma}_y + d_k^{y*}d_k^z i\cdot(-i)\hat{\sigma}_z \\
&\qquad + d_k^{z*}d_k^x(-i)\hat{\sigma}_0 + d_k^{z*}d_k^y\hat{\sigma}_z + d_k^{z*}d_k^z i^2\hat{\sigma}_y\} i\hat{\sigma}_y\Big] && (15)\\
&= \mathrm{Tr}\Big[-i\hat{\sigma}_y\{-|d_k^x|^2\hat{\sigma}_y + d_k^{x*}d_k^y\hat{\sigma}_x + i d_k^{x*}d_k^z\hat{\sigma}_0 \\
&\qquad + d_k^{y*}d_k^x\hat{\sigma}_x + |d_k^y|^2\hat{\sigma}_y + d_k^{y*}d_k^z\hat{\sigma}_z \\
&\qquad - i d_k^{z*}d_k^x\hat{\sigma}_0 + d_k^{z*}d_k^y\hat{\sigma}_z - |d_k^y|^2\hat{\sigma}_y\} i\hat{\sigma}_y\Big] && (16)\\
&= \mathrm{Tr}\Big[-|d_k^x|^2\hat{\sigma}_y - d_k^{x*}d_k^y\hat{\sigma}_x + i d_k^{x*}d_k^z\hat{\sigma}_0
\end{aligned}$$

$$-d_k^{y*}d_k^x\hat{\sigma}_x + |d_k^y|^2\hat{\sigma}_y - d_k^{y*}d_k^z\hat{\sigma}_z$$

$$-id_k^{z*}d_k^x\hat{\sigma}_0 - d_k^{z*}d_k^y\hat{\sigma}_z - |d_k^y|^2\hat{\sigma}_y\Big] \tag{17}$$

$$= \mathrm{Tr}\Big[+id_k^{x*}d_k^z\hat{\sigma}_0 - id_k^{z*}d_k^x\hat{\sigma}_0\Big] \tag{18}$$

$$= -2i(d_k^{z*}d_k^x - d_k^{x*}d_k^z) \tag{19}$$

$$= -2i[\vec{d}_k^* \times \vec{d}_k]_y. \tag{20}$$

$$\mathrm{Tr}\Big[\hat{\Delta}_k^\dagger\hat{\sigma}_z\hat{\Delta}_k\Big] = \mathrm{Tr}\Big[(\vec{d}_k\cdot\hat{\vec{\sigma}}i\hat{\sigma}_y)^\dagger\hat{\sigma}_z(\vec{d}_k\cdot\hat{\vec{\sigma}}i\hat{\sigma}_y)\Big] \tag{21}$$

$$= \mathrm{Tr}\Big[(\vec{d}_k\cdot\hat{\vec{\sigma}}i\hat{\sigma}_y)^\dagger\hat{\sigma}_z(d_k^x\hat{\sigma}_x + d_k^y\hat{\sigma}_y + d_k^z\hat{\sigma}_z)i\hat{\sigma}_y\Big] \tag{22}$$

$$= \mathrm{Tr}\Big[(\vec{d}_k\cdot\hat{\vec{\sigma}}i\hat{\sigma}_y)^\dagger\{d_k^x i\hat{\sigma}_y + d_k^y(-i\hat{\sigma}_x) + d_k^z\hat{\sigma}_0\}i\hat{\sigma}_y\Big] \tag{23}$$

$$= \mathrm{Tr}\Big[-i\hat{\sigma}_y(d_k^{x*}\hat{\sigma}_x + d_k^{y*}\hat{\sigma}_y + d_k^{z*}\hat{\sigma}_z)\{d_k^x i\hat{\sigma}_y + d_k^y(-i\hat{\sigma}_x) + d_k^z\hat{\sigma}_0\}i\hat{\sigma}_y\Big] \tag{24}$$

$$= \mathrm{Tr}\Big[-i\hat{\sigma}_y\{d_k^{x*}d_k^x i^2\hat{\sigma}_z + d_k^{x*}d_k^y(-i)\hat{\sigma}_0 + d_k^{x*}d_k^z\hat{\sigma}_z$$

$$+d_k^{y*}d_k^x i\hat{\sigma}_0 + d_k^{y*}d_k^y(-i)^2\hat{\sigma}_z + d_k^{y*}d_k^z\hat{\sigma}_y$$

$$+d_k^{z*}d_k^x i\cdot(-i)\hat{\sigma}_x + d_k^{z*}d_k^y(-i)\cdot i\hat{\sigma}_y + d_k^{z*}d_k^z\hat{\sigma}_z\}i\hat{\sigma}_y\Big] \tag{25}$$

$$= \mathrm{Tr}\Big[-i\hat{\sigma}_y\{-|d_k^x|^2\hat{\sigma}_z - id_k^{x*}d_k^y\hat{\sigma}_0 + d_k^{x*}d_k^z\hat{\sigma}_z$$

$$+id_k^{y*}d_k^x\hat{\sigma}_0 - |d_k^y|^2\hat{\sigma}_z + d_k^{y*}d_k^z\hat{\sigma}_y$$

$$+d_k^{z*}d_k^x\hat{\sigma}_x + d_k^{z*}d_k^y\hat{\sigma}_y + |d_k^z|^2\hat{\sigma}_z\}i\hat{\sigma}_y\Big] \tag{26}$$

$$= \mathrm{Tr}\Big[+|d_k^x|^2\hat{\sigma}_z - id_k^{x*}d_k^y\hat{\sigma}_0 - d_k^{x*}d_k^z\hat{\sigma}_z$$

$$+id_k^{y*}d_k^x\hat{\sigma}_0 + |d_k^y|^2\hat{\sigma}_z + d_k^{y*}d_k^z\hat{\sigma}_y$$

$$-d_k^{z*}d_k^x\hat{\sigma}_x + d_k^{z*}d_k^y\hat{\sigma}_y - |d_k^z|^2\hat{\sigma}_z\Big] \tag{27}$$

$$= \mathrm{Tr}\Big[-id_k^{x*}d_k^y\hat{\sigma}_0 + id_k^{y*}d_k^x\hat{\sigma}_0\Big] \tag{28}$$

$$= -2i(d_k^{x*}d_k^y - d_k^{y*}d_k^x) \tag{29}$$

$$= -2i[\vec{d}_k^* \times \vec{d}_k]_z. \tag{30}$$

Thus,

$$\mathrm{Tr}\Big[\hat{\Delta}_k^\dagger\hat{\vec{\sigma}}\hat{\Delta}_k\Big] = -2i(\vec{d}_k^* \times \vec{d}_k), \tag{31}$$

and then

$$\mathrm{Tr}\Big[\hat{\Delta}_k^\dagger\vec{S}\hat{\Delta}_k\Big] = \mathrm{Tr}\Big[\hat{\Delta}_k^\dagger\Big(\frac{\hbar}{2}\hat{\vec{\sigma}}\Big)\hat{\Delta}_k\Big] \tag{32}$$

$$= -i\hbar(\vec{d}_k^* \times \vec{d}_k). \tag{33}$$

On the other hand,

$$\mathrm{Tr}\Big[\hat{\Delta}_k^\dagger\hat{\Delta}_k\Big] = \mathrm{Tr}\Big[(\vec{d}_k\cdot\hat{\vec{\sigma}}i\hat{\sigma}_y)^\dagger(\vec{d}_k\cdot\hat{\vec{\sigma}}i\hat{\sigma}_y)\Big] \tag{34}$$

$$= \mathrm{Tr}\Big[-i\hat{\sigma}_y(d_k^{x*}\hat{\sigma}_x + d_k^{y*}\hat{\sigma}_y + d_k^{z*}\hat{\sigma}_z)(d_k^x\hat{\sigma}_x + d_k^y\hat{\sigma}_y + d_k^z\hat{\sigma}_z)i\hat{\sigma}_y\Big] \tag{35}$$

$$= \mathrm{Tr}\Big[-i\hat{\sigma}_y\Big\{(|d_k^x|^2 + |d_k^y|^2 + |d_k^z|^2)\hat{\sigma}_0$$

$$+i(d_k^{y*}d_k^z - d_k^{z*}d_k^y)\hat{\sigma}_x + i(d_k^{z*}d_k^x - d_k^{x*}d_k^z)\hat{\sigma}_y + i(d_k^{x*}d_k^y - d_k^{y*}d_k^z)\hat{\sigma}_z\Big\}i\hat{\sigma}_y\Big] \quad (36)$$

$$= \mathrm{Tr}\Big[|\vec{d}_k|^2\hat{\sigma}_0 - i[\vec{d}_k^* \times \vec{d}_k]_x\hat{\sigma}_x + i[\vec{d}_k^* \times \vec{d}_k]_y\hat{\sigma}_y - i[\vec{d}_k^* \times \vec{d}_k]_z\hat{\sigma}_z\Big] \quad (37)$$

$$= \mathrm{Tr}\Big[|\vec{d}_k|^2\hat{\sigma}_0\Big] \quad (38)$$

$$= 2|\vec{d}_k|^2 \quad (39)$$

We define $|\vec{d}| = \sqrt{|\vec{d}|^2}$ as

$$|\vec{d}|^2 \equiv \int \frac{d\Omega_k}{4\pi}\frac{1}{2}\mathrm{Tr}\Big[\hat{\Delta}_k^\dagger\hat{\Delta}_k\Big] = \int \frac{d\Omega_k}{4\pi}|\vec{d}_k|^2. \quad (40)$$

Then,

$$\int \frac{d\Omega_k}{4\pi}\mathrm{Tr}\Big[\hat{\Delta}_k^\dagger\hat{\Delta}_k\Big] = 2\int \frac{d\Omega_k}{4\pi}|\vec{d}_k|^2 = 2|\vec{d}|^2. \quad (41)$$

Hence, from Eqs. (33) and (41),

$$\langle\vec{S}\rangle \equiv \frac{\int \frac{d\Omega_k}{4\pi}\mathrm{Tr}[\hat{\Delta}_k^\dagger\vec{S}\hat{\Delta}_k]}{\int \frac{d\Omega_k}{4\pi}\mathrm{Tr}[\hat{\Delta}_k^\dagger\hat{\Delta}_k]} \quad (42)$$

$$= \frac{\int \frac{d\Omega_k}{4\pi}[-\hbar i(\vec{d}_k^* \times \vec{d}_k)]}{2|\vec{d}|^2} \quad (43)$$

$$= -i\frac{\hbar}{2}\int \frac{d\Omega_k}{4\pi}\hat{d}_k^* \times \hat{d}_k, \quad (44)$$

where $\hat{d}_k \equiv \vec{d}_k/|\vec{d}|$.

We will take the spherical coordinates such that $(\hat{k}_x, \hat{k}_y, \hat{k}_z) = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$.

For the A-phase $\hat{d}_k = \sqrt{\frac{3}{2}}(0, 0, \hat{k}_x + i\hat{k}_y)$,

$$\hat{d}_k^* \times \hat{d}_k = \sqrt{\frac{3}{2}}(0, 0, \hat{k}_x - i\hat{k}_y) \times \sqrt{\frac{3}{2}}(0, 0, \hat{k}_x + i\hat{k}_y) \quad (45)$$

$$= 0, \quad (46)$$

and then

$$\langle\vec{S}\rangle = -i\frac{\hbar}{2}\int \frac{d\Omega_k}{4\pi}\hat{d}_k^* \times \hat{d}_k \quad (47)$$

$$= 0. \quad (48)$$

For the A$_1$-phase $\hat{d}_k = \sqrt{\frac{3}{4}}\Big(\hat{k}_y + i\hat{k}_z,\ i(\hat{k}_y + i\hat{k}_z),\ 0\Big)$,

$$\hat{d}_k^* \times \hat{d}_k = \sqrt{\frac{3}{4}}\Big(\hat{k}_y - i\hat{k}_z,\ -i(\hat{k}_y - i\hat{k}_z),\ 0\Big) \times \sqrt{\frac{3}{4}}\Big(\hat{k}_y + i\hat{k}_z,\ i(\hat{k}_y + i\hat{k}_z),\ 0\Big) \quad (49)$$

$$= \frac{3}{4}\Big(0, 0, (\hat{k}_y - i\hat{k}_z)i(\hat{k}_y + i\hat{k}_z) - \{-i(\hat{k}_y - i\hat{k}_z)\}(\hat{k}_y + i\hat{k}_z)\Big) \quad (50)$$

$$= \frac{3}{4}\Big(0, 0, 2i(\hat{k}_y^2 + \hat{k}_z^2)\Big), \quad (51)$$

and then

$$\begin{aligned}
\langle\vec{S}\rangle &= -i\frac{\hbar}{2}\int\frac{d\Omega_k}{4\pi}\hat{d}_k^*\times\hat{d}_k && (52)\\
&= -i\frac{\hbar}{2}\frac{3}{4}\int\frac{d\Omega_k}{4\pi}\Big(0,0,2i(\hat{k}_y^2+\hat{k}_z^2)\Big) && (53)\\
&= -i\frac{\hbar}{2}\frac{3}{4}\frac{1}{4\pi}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\sin\theta\Big(0,0,2i(\sin^2\phi\sin^2\theta+\cos^2\theta)\Big) && (54)\\
&= -i\frac{\hbar}{2}\frac{3}{4}\frac{1}{4\pi}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\Big(0,0,2i(\sin^2\phi\sin^3\theta+\sin\theta\cos^2\theta)\Big) && (55)\\
&= -i\frac{\hbar}{2}\frac{3}{4}\frac{1}{2}\int_0^{\pi}d\theta\Big(0,0,2i(\frac{1}{2}\sin^3\theta+\sin\theta\cos^2\theta)\Big) && (56)\\
&= -i\frac{\hbar}{2}\frac{3}{4}\frac{1}{2}\Big(0,0,2i(\frac{1}{2}\frac{4}{3}+\frac{2}{3})\Big) && (57)\\
&= -i\frac{\hbar}{2}\frac{3}{4}\frac{1}{2}\Big(0,0,2i\frac{4}{3}\Big) && (58)\\
&= \Big(0,0,\frac{1}{2}\hbar\Big). && (59)
\end{aligned}$$

For the B-phase $\hat{d}_k=(\hat{k}_x,\hat{k}_y,\hat{k}_z)$,

$$\begin{aligned}
\hat{d}_k^*\times\hat{d}_k &= (\hat{k}_x,\hat{k}_y,\hat{k}_z)\times(\hat{k}_x,\hat{k}_y,\hat{k}_z) && (60)\\
&= 0, && (61)
\end{aligned}$$

and then

$$\begin{aligned}
\langle\vec{S}\rangle &= -i\frac{\hbar}{2}\int\frac{d\Omega_k}{4\pi}\hat{d}_k^*\times\hat{d}_k && (62)\\
&= 0. && (63)
\end{aligned}$$

5.2 Let us consider $\vec{L} = -i\hbar(\hat{k} \times \vec{\nabla}_{\hat{k}})$.

$$
\begin{aligned}
\mathrm{Tr}\Big[\hat{\Delta}_k^\dagger \vec{L}\hat{\Delta}_k\Big] &= \mathrm{Tr}\Big[(\vec{d}_k \cdot \hat{\vec{\sigma}} i\hat{\sigma}_y)^\dagger \vec{L}(\vec{d}_k \cdot \hat{\vec{\sigma}} i\hat{\sigma}_y)\Big] && (64)\\
&= \mathrm{Tr}\Big[(\vec{d}_k \cdot \hat{\vec{\sigma}} i\hat{\sigma}_y)^\dagger \vec{L}(d_k^x\hat{\sigma}_x + d_k^y\hat{\sigma}_y + d_k^z\hat{\sigma}_z)i\hat{\sigma}_y\Big] && (65)\\
&= \mathrm{Tr}\Big[-i\hat{\sigma}_y(d_k^{x*}\hat{\sigma}_x + d_k^{y*}\hat{\sigma}_y + d_k^{z*}\hat{\sigma}_z)\\
&\qquad \times(\vec{L}d_k^x\hat{\sigma}_x + \vec{L}d_k^y\hat{\sigma}_y + \vec{L}d_k^z\hat{\sigma}_z)i\hat{\sigma}_y\Big] && (66)\\
&= \mathrm{Tr}\Big[-i\hat{\sigma}_y\Big\{(d_k^{x*}\vec{L}d_k^x + d_k^{y*}\vec{L}d_k^y + d_k^{z*}\vec{L}d_k^z)\hat{\sigma}_0\\
&\qquad\qquad +i(d_k^{y*}\vec{L}d_k^z - d_k^{z*}\vec{L}d_k^y)\hat{\sigma}_x\\
&\qquad\qquad +i(d_k^{z*}\vec{L}d_k^x - d_k^{x*}\vec{L}d_k^z)\hat{\sigma}_y\\
&\qquad\qquad +i(d_k^{x*}\vec{L}d_k^y - d_k^{y*}\vec{L}d_k^z)\hat{\sigma}_z\Big\}i\hat{\sigma}_y\Big] && (67)\\
&= \mathrm{Tr}\Big[(d_k^{x*}\vec{L}d_k^x + d_k^{y*}\vec{L}d_k^y + d_k^{z*}\vec{L}d_k^z)\hat{\sigma}_0\\
&\qquad -i(d_k^{y*}\vec{L}d_k^z - d_k^{z*}\vec{L}d_k^y)\hat{\sigma}_x\\
&\qquad +i(d_k^{z*}\vec{L}d_k^x - d_k^{x*}\vec{L}d_k^z)\hat{\sigma}_y\\
&\qquad -i(d_k^{x*}\vec{L}d_k^y - d_k^{y*}\vec{L}d_k^z)\hat{\sigma}_z\Big] && (68)\\
&= \mathrm{Tr}\Big[(d_k^{x*}\vec{L}d_k^x + d_k^{y*}\vec{L}d_k^y + d_k^{z*}\vec{L}d_k^z)\hat{\sigma}_0\Big] && (69)\\
&= 2(d_k^{x*}\vec{L}d_k^x + d_k^{y*}\vec{L}d_k^y + d_k^{z*}\vec{L}d_k^z) && (70)\\
&= 2\vec{d}_k^* \cdot [\vec{L}\vec{d}_k] && (71)\\
&= -2i\hbar\vec{d}_k^* \cdot [(\hat{k} \times \vec{\nabla}_{\hat{k}})\vec{d}_k]. && (72)
\end{aligned}
$$

Hence, from Eqs. (41) and (72),

$$
\begin{aligned}
\langle\vec{L}\rangle &\equiv \frac{\int \frac{d\Omega_k}{4\pi}\mathrm{Tr}[\hat{\Delta}_k^\dagger\vec{L}\hat{\Delta}_k]}{\int \frac{d\Omega_k}{4\pi}\mathrm{Tr}[\hat{\Delta}_k^\dagger\hat{\Delta}_k]} && (73)\\
&= \frac{\int \frac{d\Omega_k}{4\pi}(-2i\hbar)\vec{d}_k^* \cdot [(\hat{k} \times \vec{\nabla}_{\hat{k}})\vec{d}_k]}{2|\vec{d}|^2} && (74)\\
&= -i\hbar\int \frac{d\Omega_k}{4\pi}\hat{d}_k^* \cdot [(\hat{k} \times \vec{\nabla}_{\hat{k}})\hat{d}_k]. && (75)
\end{aligned}
$$

We will take the spherical coordinates such that $(\hat{k}_x, \hat{k}_y, \hat{k}_z) = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$. For the A-phase $\hat{d}_k = \sqrt{\frac{3}{2}}(0, 0, \hat{k}_x + i\hat{k}_y)$,

$$
\begin{aligned}
\hat{d}_k^* \cdot [(\hat{k} \times \vec{\nabla}_{\hat{k}})_x\hat{d}_k] &= \hat{d}_k^* \cdot \sqrt{\frac{3}{2}}\Big(0, 0, (\hat{k} \times \vec{\nabla}_{\hat{k}})_x(\hat{k}_x + i\hat{k}_y)\Big) && (76)\\
&= \sqrt{\frac{3}{2}}\hat{d}_k^* \cdot \left(0, 0, \Big(\hat{k}_y\frac{\partial}{\partial\hat{k}_z} - \hat{k}_z\frac{\partial}{\partial\hat{k}_y}\Big)(\hat{k}_x + i\hat{k}_y)\right) && (77)\\
&= \sqrt{\frac{3}{2}}\hat{d}_k^* \cdot (0, 0, -i\hat{k}_z) && (78)\\
&= \frac{3}{2}(0, 0, \hat{k}_x - i\hat{k}_y) \cdot (0, 0, -i\hat{k}_z) && (79)\\
&= \frac{3}{2}(-i\hat{k}_x\hat{k}_z - \hat{k}_y\hat{k}_z), && (80)
\end{aligned}
$$

$$\hat{d}_k^* \cdot [(\hat{k}\times\vec{\nabla}_{\hat{k}})_y \hat{d}_k] = \hat{d}_k^* \cdot \sqrt{\frac{3}{2}}\Big(0,0,(\hat{k}\times\vec{\nabla}_{\hat{k}})_y(\hat{k}_x+i\hat{k}_y)\Big) \quad (81)$$

$$= \sqrt{\frac{3}{2}}\hat{d}_k^* \cdot \left(0,0,\big(\hat{k}_z\frac{\partial}{\partial\hat{k}_x}-\hat{k}_x\frac{\partial}{\partial\hat{k}_z}\big)(\hat{k}_x+i\hat{k}_y)\right) \quad (82)$$

$$= \sqrt{\frac{3}{2}}\hat{d}_k^* \cdot (0,0,\hat{k}_z) \quad (83)$$

$$= \frac{3}{2}(0,0,\hat{k}_x-i\hat{k}_y)\cdot(0,0,\hat{k}_z) \quad (84)$$

$$= \frac{3}{2}(\hat{k}_x\hat{k}_z-i\hat{k}_y\hat{k}_z), \quad (85)$$

$$\hat{d}_k^* \cdot [(\hat{k}\times\vec{\nabla}_{\hat{k}})_z \hat{d}_k] = \hat{d}_k^* \cdot \sqrt{\frac{3}{2}}\Big(0,0,(\hat{k}\times\vec{\nabla}_{\hat{k}})_z(\hat{k}_x+i\hat{k}_y)\Big) \quad (86)$$

$$= \sqrt{\frac{3}{2}}\hat{d}_k^* \cdot \left(0,0,\big(\hat{k}_x\frac{\partial}{\partial\hat{k}_y}-\hat{k}_y\frac{\partial}{\partial\hat{k}_x}\big)(\hat{k}_x+i\hat{k}_y)\right) \quad (87)$$

$$= \sqrt{\frac{3}{2}}\hat{d}_k^* \cdot (0,0,i\hat{k}_x-\hat{k}_y) \quad (88)$$

$$= \frac{3}{2}(0,0,\hat{k}_x-i\hat{k}_y)\cdot\Big(0,0,i(\hat{k}_x+i\hat{k}_y)\Big) \quad (89)$$

$$= i\frac{3}{2}(\hat{k}_x^2+\hat{k}_y^2), \quad (90)$$

and then

$$\langle\vec{L}\rangle = -i\hbar\int\frac{d\Omega_k}{4\pi}\hat{d}_k^*\cdot[(\hat{k}\times\vec{\nabla}_{\hat{k}})\hat{d}_k] \quad (91)$$

$$= -i\hbar\frac{3}{2}\int\frac{d\Omega_k}{4\pi}\Big(-i\hat{k}_x\hat{k}_z-\hat{k}_y\hat{k}_z,\ \hat{k}_x\hat{k}_z-i\hat{k}_y\hat{k}_z,\ i(\hat{k}_x^2+\hat{k}_y^2)\Big) \quad (92)$$

$$= -i\hbar\frac{3}{2}\frac{1}{4\pi}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\sin\theta\Big(-i\cos\phi\sin\theta\cos\theta-\sin\phi\sin\theta\cos\theta,$$
$$\cos\phi\sin\theta\cos\theta-i\sin\phi\sin\theta\cos\theta,$$
$$i(\cos^2\phi\sin^2\theta+\sin^2\phi\sin^2\theta)\Big) \quad (93)$$

$$= -i\hbar\frac{3}{2}\frac{1}{4\pi}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\sin\theta\Big(-i\cos\phi\sin\theta\cos\theta-\sin\phi\sin\theta\cos\theta,$$
$$\cos\phi\sin\theta\cos\theta-i\sin\phi\sin\theta\cos\theta,$$
$$i\sin^2\theta\Big) \quad (94)$$

$$= -i\hbar\frac{3}{2}\frac{1}{2}\int_0^{\pi}d\theta\sin\theta\Big(0,0,i\sin^2\theta\Big) \quad (95)$$

$$= -i\hbar\frac{3}{2}\frac{i}{2}\int_0^{\pi}d\theta\Big(0,0,\sin^3\theta\Big) \quad (96)$$

$$= -i\hbar\frac{3}{2}\frac{i}{2}\Big(0,0,\frac{4}{3}\Big) \quad (97)$$

$$= \Big(0,0,\hbar\Big). \quad (98)$$

For the A₁-phase $\hat{d}_k = \sqrt{\frac{3}{4}}\Big(\hat{k}_y+i\hat{k}_z,\ i(\hat{k}_y+i\hat{k}_z),\ 0\Big)$,

$$\hat{d}_k^* \cdot [(\hat{k}\times\vec{\nabla}_{\hat{k}})_x \hat{d}_k] = \hat{d}_k^* \cdot (\hat{k}\times\vec{\nabla}_{\hat{k}})_x\sqrt{\frac{3}{4}}\Big(\hat{k}_y+i\hat{k}_z,\ i(\hat{k}_y+i\hat{k}_z),\ 0\Big) \quad (99)$$

$$= \sqrt{\frac{3}{4}}\hat{d}_k^* \cdot \Big(\hat{k}_y \frac{\partial}{\partial \hat{k}_z} - \hat{k}_z \frac{\partial}{\partial \hat{k}_y}\Big)\Big(\hat{k}_y + i\hat{k}_z, i(\hat{k}_y + i\hat{k}_z), 0\Big) \quad (100)$$

$$= \sqrt{\frac{3}{4}}\hat{d}_k^* \cdot \Big(-\hat{k}_z + i\hat{k}_y, i(-\hat{k}_z + i\hat{k}_y), 0\Big) \quad (101)$$

$$= \frac{3}{4}\Big(\hat{k}_y - i\hat{k}_z, -i(\hat{k}_y - i\hat{k}_z), 0\Big) \cdot \Big(-\hat{k}_z + i\hat{k}_y, i(-\hat{k}_z + i\hat{k}_y), 0\Big) \quad (102)$$

$$= \frac{3}{4}\Big((\hat{k}_y - i\hat{k}_z)(-\hat{k}_z + i\hat{k}_y) + \{-i(\hat{k}_y - i\hat{k}_z)\}i(-\hat{k}_z + i\hat{k}_y)\Big) \quad (103)$$

$$= \frac{3}{4}\Big(i(\hat{k}_y - i\hat{k}_z)(\hat{k}_y + i\hat{k}_z) + i(\hat{k}_y - i\hat{k}_z)(\hat{k}_y + i\hat{k}_z)\Big) \quad (104)$$

$$= i\frac{3}{2}(\hat{k}_y^2 + \hat{k}_z^2), \quad (105)$$

$$\hat{d}_k^* \cdot [(\hat{k} \times \vec{\nabla}_{\hat{k}})_y \hat{d}_k] = \hat{d}_k^* \cdot (\hat{k} \times \vec{\nabla}_{\hat{k}})_y \sqrt{\frac{3}{4}}\Big(\hat{k}_y + i\hat{k}_z, i(\hat{k}_y + i\hat{k}_z), 0\Big) \quad (106)$$

$$= \sqrt{\frac{3}{4}}\hat{d}_k^* \cdot \Big(\hat{k}_z \frac{\partial}{\partial \hat{k}_x} - \hat{k}_x \frac{\partial}{\partial \hat{k}_z}\Big)\Big(\hat{k}_y + i\hat{k}_z, i(\hat{k}_y + i\hat{k}_z), 0\Big) \quad (107)$$

$$= \sqrt{\frac{3}{4}}\hat{d}_k^* \cdot \Big(-i\hat{k}_x, i(-i\hat{k}_x), 0\Big) \quad (108)$$

$$= \frac{3}{4}\Big(\hat{k}_y - i\hat{k}_z, -i(\hat{k}_y - i\hat{k}_z), 0\Big) \cdot \Big(-i\hat{k}_x, \hat{k}_x, 0\Big) \quad (109)$$

$$= \frac{3}{4}\Big((\hat{k}_y - i\hat{k}_z)(-i\hat{k}_x) + \{-i(\hat{k}_y - i\hat{k}_z)\}\hat{k}_x\Big) \quad (110)$$

$$= -i\frac{3}{2}(\hat{k}_x\hat{k}_y - i\hat{k}_x\hat{k}_z), \quad (111)$$

$$\hat{d}_k^* \cdot [(\hat{k} \times \vec{\nabla}_{\hat{k}})_z \hat{d}_k] = \hat{d}_k^* \cdot (\hat{k} \times \vec{\nabla}_{\hat{k}})_z \sqrt{\frac{3}{4}}\Big(\hat{k}_y + i\hat{k}_z, i(\hat{k}_y + i\hat{k}_z), 0\Big) \quad (112)$$

$$= \sqrt{\frac{3}{4}}\hat{d}_k^* \cdot \Big(\hat{k}_x \frac{\partial}{\partial \hat{k}_y} - \hat{k}_y \frac{\partial}{\partial \hat{k}_x}\Big)\Big(\hat{k}_y + i\hat{k}_z, i(\hat{k}_y + i\hat{k}_z), 0\Big) \quad (113)$$

$$= \sqrt{\frac{3}{4}}\hat{d}_k^* \cdot \Big(\hat{k}_x, i\hat{k}_x, 0\Big) \quad (114)$$

$$= \frac{3}{4}\Big(\hat{k}_y - i\hat{k}_z, -i(\hat{k}_y - i\hat{k}_z), 0\Big) \cdot \Big(\hat{k}_x, i\hat{k}_x, 0\Big) \quad (115)$$

$$= \frac{3}{4}\Big((\hat{k}_y - i\hat{k}_z)\hat{k}_x + \{-i(\hat{k}_y - i\hat{k}_z)\}i\hat{k}_x\Big) \quad (116)$$

$$= \frac{3}{2}(\hat{k}_x\hat{k}_y - i\hat{k}_x\hat{k}_z), \quad (117)$$

and then

$$\langle \vec{L} \rangle = -i\hbar \int \frac{d\Omega_k}{4\pi}\hat{d}_k^* \cdot [(\hat{k} \times \vec{\nabla}_{\hat{k}})\hat{d}_k] \quad (118)$$

$$= -i\hbar \int \frac{d\Omega_k}{4\pi}\Big(i\frac{3}{2}(\hat{k}_y^2 + \hat{k}_z^2),\ -i\frac{3}{2}(\hat{k}_x\hat{k}_y - i\hat{k}_x\hat{k}_z),\ \frac{3}{2}(\hat{k}_x\hat{k}_y - i\hat{k}_x\hat{k}_z)\Big) \quad (119)$$

$$
\begin{aligned}
&= -i\hbar\frac{3}{2}\frac{1}{4\pi}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\sin\theta\Big(i(\sin^2\phi\sin^2\theta+\cos^2\theta),\\
&\qquad -i(\cos\phi\sin\theta\sin\phi\sin\theta - i\cos\phi\sin\theta\cos\theta),\\
&\qquad i(\cos\phi\sin\theta\sin\phi\sin\theta - i\cos\phi\sin\theta\cos\theta)\Big) \qquad (120)\\
&= -i\hbar\frac{3}{2}\frac{1}{2}\int_0^{\pi}d\theta\sin\theta\Big(i(\frac{1}{2}\sin^2\theta+\cos^2\theta),0,0\Big) \qquad (121)\\
&= -i\hbar\frac{3}{2}\frac{i}{2}\int_0^{\pi}d\theta\Big(\frac{1}{2}\sin^3\theta+\sin\theta\cos^2\theta,0,0\Big) \qquad (122)\\
&= -i\hbar\frac{3}{2}\frac{i}{2}\Big(\frac{1}{2}\frac{4}{3}+\frac{2}{3},0,0\Big) \qquad (123)\\
&= -i\hbar\frac{3}{2}\frac{i}{2}\Big(\frac{4}{3},0,0\Big) \qquad (124)\\
&= \Big(\hbar,0,0\Big). \qquad (125)
\end{aligned}
$$

For the B-phase $\hat{d}_k = (\hat{k}_x, \hat{k}_y, \hat{k}_z)$,

$$
\begin{aligned}
\hat{d}_k^*\cdot[(\hat{k}\times\vec{\nabla}_{\hat{k}})_x\hat{d}_k] &= \hat{d}_k^*\cdot(\hat{k}\times\vec{\nabla}_{\hat{k}})_x\Big(\hat{k}_x,\hat{k}_y,\hat{k}_z\Big) \qquad (126)\\
&= \hat{d}_k^*\cdot\Big(\hat{k}_y\frac{\partial}{\partial\hat{k}_z}-\hat{k}_z\frac{\partial}{\partial\hat{k}_y}\Big)\Big(\hat{k}_x,\hat{k}_y,\hat{k}_z\Big) \qquad (127)\\
&= \hat{d}_k^*\cdot(0,-\hat{k}_z,\hat{k}_y) \qquad (128)\\
&= (\hat{k}_x,\hat{k}_y,\hat{k}_z)\cdot(0,-\hat{k}_z,\hat{k}_y) \qquad (129)\\
&= 0, \qquad (130)
\end{aligned}
$$

$$
\begin{aligned}
\hat{d}_k^*\cdot[(\hat{k}\times\vec{\nabla}_{\hat{k}})_y\hat{d}_k] &= \hat{d}_k^*\cdot(\hat{k}\times\vec{\nabla}_{\hat{k}})_y\Big(\hat{k}_x,\hat{k}_y,\hat{k}_z\Big) \qquad (131)\\
&= \hat{d}_k^*\cdot\Big(\hat{k}_z\frac{\partial}{\partial\hat{k}_x}-\hat{k}_x\frac{\partial}{\partial\hat{k}_z}\Big)\Big(\hat{k}_x,\hat{k}_y,\hat{k}_z\Big) \qquad (132)\\
&= \hat{d}_k^*\cdot(\hat{k}_z,0,-\hat{k}_x) \qquad (133)\\
&= (\hat{k}_x,\hat{k}_y,\hat{k}_z)\cdot(\hat{k}_z,0,-\hat{k}_x) \qquad (134)\\
&= 0, \qquad (135)
\end{aligned}
$$

$$
\begin{aligned}
\hat{d}_k^*\cdot[(\hat{k}\times\vec{\nabla}_{\hat{k}})_z\hat{d}_k] &= \hat{d}_k^*\cdot(\hat{k}\times\vec{\nabla}_{\hat{k}})_z\Big(\hat{k}_x,\hat{k}_y,\hat{k}_z\Big) \qquad (136)\\
&= \hat{d}_k^*\cdot\Big(\hat{k}_x\frac{\partial}{\partial\hat{k}_y}-\hat{k}_y\frac{\partial}{\partial\hat{k}_x}\Big)\Big(\hat{k}_x,\hat{k}_y,\hat{k}_z\Big) \qquad (137)\\
&= \hat{d}_k^*\cdot(-\hat{k}_y,\hat{k}_x,0) \qquad (138)\\
&= (\hat{k}_x,\hat{k}_y,\hat{k}_z)\cdot(-\hat{k}_y,\hat{k}_x,0) \qquad (139)\\
&= 0, \qquad (140)
\end{aligned}
$$

and then

$$
\begin{aligned}
\langle\vec{L}\rangle &= -i\hbar\int\frac{d\Omega_k}{4\pi}\hat{d}_k^*\cdot[(\hat{k}\times\vec{\nabla}_{\hat{k}})\hat{d}_k] \qquad (141)\\
&= -i\hbar\int\frac{d\Omega_k}{4\pi}\Big(0,\ 0,\ 0\Big) \qquad (142)\\
&= \Big(0,0,0\Big). \qquad (143)
\end{aligned}
$$

In summary:

- For the A-phase $\hat{d}_k = \sqrt{\frac{3}{2}}(0, 0, \hat{k}_x + i\hat{k}_y)$,

$$\langle \vec{S} \rangle = 0 \qquad \text{and} \qquad \langle \vec{L} \rangle = (0, 0, \hbar). \tag{144}$$

- For the A$_1$-phase $\hat{d}_k = \sqrt{\frac{3}{4}}\big(\hat{k}_y + i\hat{k}_z, i(\hat{k}_y + i\hat{k}_z), 0\big)$,

$$\langle \vec{S} \rangle = \Big(0, 0, \frac{1}{2}\hbar\Big) \qquad \text{and} \qquad \langle \vec{L} \rangle = (\hbar, 0, 0). \tag{145}$$

- For the B-phase $\hat{d}_k = (\hat{k}_x, \hat{k}_y, \hat{k}_z)$,

$$\langle \vec{S} \rangle = 0 \qquad \text{and} \qquad \langle \vec{L} \rangle = 0. \tag{146}$$

Appendix :
In the case of the A$_1$-phase $\hat{d}_k = \sqrt{\frac{3}{4}}\big(\hat{k}_y + i\hat{k}_z, i(\hat{k}_y + i\hat{k}_z), 0\big) = \sqrt{\frac{3}{4}}(\hat{k}_y + i\hat{k}_z)\big(1, i, 0\big)$,

$$\begin{aligned}
\hat{\Delta}_k &= \vec{d}_k \cdot \hat{\vec{\sigma}} i\hat{\sigma}_y && (147)\\
&= |\vec{d}|\hat{d}_k \cdot \hat{\vec{\sigma}} i\hat{\sigma}_y && (148)\\
&= |\vec{d}| \begin{pmatrix} -\hat{d}_k^x + i\hat{d}_k^y & \hat{d}_k^z \\ \hat{d}_k^z & \hat{d}_k^x + i\hat{d}_k^y \end{pmatrix} && (149)\\
&= \sqrt{\frac{3}{4}}|\vec{d}|(\hat{k}_y + i\hat{k}_z) \begin{pmatrix} -1 + i\cdot i & 0 \\ 0 & 1 + i\cdot i \end{pmatrix} && (150)\\
&= -\sqrt{3}|\vec{d}|(\hat{k}_y + i\hat{k}_z) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. && (151)
\end{aligned}$$

This means

$$\hat{\Delta}_k = \begin{pmatrix} \Delta_{\uparrow\uparrow} & \Delta_{\uparrow\downarrow} \\ \Delta_{\downarrow\uparrow} & \Delta_{\downarrow\downarrow} \end{pmatrix} = \begin{pmatrix} \Delta_{\uparrow\uparrow} & 0 \\ 0 & 0 \end{pmatrix}, \tag{152}$$

namely $\Delta_{\uparrow\uparrow} \neq 0$ and $\Delta_{\downarrow\downarrow} = 0$, (also, $\Delta_{\uparrow\downarrow} = \Delta_{\downarrow\uparrow} = 0$) for the A$_1$-phase.

Also, it is noticed that the A$_1$-phase state is not a unitary state because $\hat{\Delta}_k\hat{\Delta}_k^\dagger$ is not proportional to the unit matrix $\hat{\sigma}_0$ (see Eq. (152)). Such a state is called nonunitary.

In general, $\hat{\Delta}_k\hat{\Delta}_k^\dagger$ can be written as $\hat{\Delta}_k\hat{\Delta}_k^\dagger = |\vec{d}_k|^2\hat{\sigma}_0 + i(\vec{d}_k \times \vec{d}_k^*) \cdot \hat{\vec{\sigma}}$. Then, the nonunitary means $\vec{d}_k \times \vec{d}_k^* \neq 0$, namely the vector $\vec{d}_k^*$ is not parallel to $\vec{d}_k$. Indeed, it is the case for the A$_1$-phase $\vec{d}_k \propto (1, i, 0)$. The vector $\vec{d}_k^* \propto (1, -i, 0)$ is not parallel to $\vec{d}_k \propto (1, i, 0)$.

Therefore, $\langle \vec{S} \rangle$ is finite for the A$_1$-phase (see Eq. (44)).

On the other hand, $\vec{d}_k \propto (0, 0, 1)$ for the A-phase and $\vec{d}_k \propto (\hat{k}_x, \hat{k}_y, \hat{k}_z)$ for the B-phase. For those two phases, $\langle \vec{S} \rangle = 0$ because $\vec{d}_k^* \parallel \vec{d}_k$ (i.e., unitary states).