Unkonventionelle Supraleitung WS 05/06
Lösungen zur Serie 6

6.1 For the spin-triplet unitary states, the energy spectrum E_k (≥ 0) is given by

$$E_k = \sqrt{\xi_k^2 + |\Delta_k|^2}, \qquad \text{with} \qquad |\Delta_k|^2 = \frac{1}{2}\mathrm{Tr}[\hat{\Delta}_k \hat{\Delta}_k^\dagger]. \tag{1}$$

Therefore,

$$E_k^2 = \xi_k^2 + |\Delta_k|^2, \tag{2}$$

$$2E_k dE_k = 2\xi_k d\xi_k, \tag{3}$$

$$d\xi_k = \frac{E_k}{\xi_k} dE_k. \tag{4}$$

$$= \frac{E_k}{\mathrm{Re}\{\pm\sqrt{E_k^2 - |\Delta_k|^2}\}} dE_k, \tag{5}$$

where $\xi_k = \mathrm{Re}\{\pm\sqrt{E_k^2 - |\Delta_k|^2}\}$, and we have taken the real part because ξ_k is real.

The density of states $N(E)$ (for $E \geq 0$) is

$$N(E) = \frac{1}{V}\sum_k \delta(E_k - E) \tag{6}$$

$$= \frac{1}{V}\sum_k \delta(\sqrt{\xi_k^2 + |\Delta_k|^2} - E) \tag{7}$$

$$= N_0 \int \frac{d\Omega_k}{4\pi} \int_0^\infty d\xi_k \delta(\sqrt{\xi_k^2 + |\Delta_k|^2} - E), \tag{8}$$

$$= N_0 \int \frac{d\Omega_k}{4\pi} \int_0^\infty dE_k \frac{E_k}{\mathrm{Re}\{+\sqrt{E_k^2 - |\Delta_k|^2}\}} \delta(E_k - E), \qquad (\xi_k \geq 0) \tag{9}$$

$$= N_0 \int \frac{d\Omega_k}{4\pi} \int_0^\infty dE_k \mathrm{Re} \frac{E_k}{\sqrt{E_k^2 - |\Delta_k|^2}} \delta(E_k - E) \tag{10}$$

$$= N_0 \int \frac{d\Omega_k}{4\pi} \mathrm{Re} \frac{E}{\sqrt{E^2 - |\Delta_k|^2}}. \tag{11}$$

6.2 For the spin-triplet unitary states,

$$|\Delta_k|^2 = \frac{1}{2}\mathrm{Tr}[\hat{\Delta}_k \hat{\Delta}_k^\dagger] \tag{12}$$

$$= \frac{1}{2}\mathrm{Tr}\Big[(\vec{d}_k \cdot \hat{\vec{\sigma}} i\hat{\sigma}_y)(\vec{d}_k \cdot \hat{\vec{\sigma}} i\hat{\sigma}_y)^\dagger\Big] \tag{13}$$

$$= \frac{1}{2}\mathrm{Tr}\Big[(\vec{d}_k \cdot \hat{\vec{\sigma}} i\hat{\sigma}_y)(-i\hat{\sigma}_y \vec{d}_k^* \cdot \hat{\vec{\sigma}})\Big] \tag{14}$$

$$= \frac{1}{2}\mathrm{Tr}\Big[(\vec{d}_k \cdot \hat{\vec{\sigma}})(\vec{d}_k^* \cdot \hat{\vec{\sigma}})\Big] \tag{15}$$

$$\begin{aligned}
&= \frac{1}{2}\mathrm{Tr}\Big[(d_k^x\hat\sigma_x + d_k^y\hat\sigma_y + d_k^z\hat\sigma_z)(d_k^{x*}\hat\sigma_x + d_k^{y*}\hat\sigma_y + d_k^{z*}\hat\sigma_z)\Big] && (16)\\
&= \frac{1}{2}\mathrm{Tr}\Big[(|d_k^x|^2 + |d_k^y|^2 + |d_k^z|^2)\hat\sigma_0 \\
&\qquad + i(d_k^y d_k^{z*} - d_k^z d_k^{y*})\hat\sigma_x + i(d_k^z d_k^{x*} - d_k^x d_k^{z*})\hat\sigma_y + i(d_k^x d_k^{y*} - d_k^y d_k^{z*})\hat\sigma_z\Big] && (17)\\
&= \frac{1}{2}\mathrm{Tr}\Big[(|d_k^x|^2 + |d_k^y|^2 + |d_k^z|^2)\hat\sigma_0\Big] && (18)\\
&= |d_k^x|^2 + |d_k^y|^2 + |d_k^z|^2. && (19)
\end{aligned}$$

(1) For the ABM state $\vec{d}_k = \Delta_0(0,0,\hat{k}_x + i\hat{k}_y)$ [which has point gap nodes at $\hat{k}_x = \hat{k}_y = 0$ (i.e., at the points $\theta = 0,\ \pi$)],

$$\begin{aligned}
|\Delta_k|^2 &= |d_k^x|^2 + |d_k^y|^2 + |d_k^z|^2 && (20)\\
&= |\Delta_0|^2(\hat{k}_x + i\hat{k}_y)(\hat{k}_x - i\hat{k}_y) && (21)\\
&= |\Delta_0|^2(\hat{k}_x^2 + \hat{k}_y^2) && (22)\\
&= |\Delta_0|^2\Big((\cos\phi\sin\theta)^2 + (\sin\phi\sin\theta)^2\Big) && (23)\\
&= |\Delta_0|^2\sin^2\theta. && (24)
\end{aligned}$$

Here, $(\hat{k}_x, \hat{k}_y, \hat{k}_z) = (\cos\phi\sin\theta,\ \sin\phi\sin\theta,\ \cos\theta)$.

Inserting this into the expression for $N(E)$ (for $E \geq 0$) in Eq. (11),

$$\begin{aligned}
N(E) &= N_0\int\frac{d\Omega_k}{4\pi}\mathrm{Re}\frac{E}{\sqrt{E^2 - |\Delta_k|^2}} && (25)\\
&= N_0 E\cdot\mathrm{Re}\int\frac{d\Omega_k}{4\pi}\frac{1}{\sqrt{E^2 - |\Delta_0|^2(\hat{k}_x^2 + \hat{k}_y^2)}} && (26)\\
&= N_0\frac{E}{4\pi}\mathrm{Re}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\sin\theta\frac{1}{\sqrt{E^2 - |\Delta_0|^2\sin^2\theta}} && (27)\\
&= N_0\frac{E}{2}\mathrm{Re}\int_0^{\pi}d\theta\sin\theta\frac{1}{\sqrt{E^2 - |\Delta_0|^2\sin^2\theta}} && (28)\\
&= N_0\frac{E}{2}\mathrm{Re}\int_{-1}^{1}dt\frac{1}{\sqrt{E^2 - |\Delta_0|^2(1-t^2)}}, \qquad (t = \cos\theta) && (29)\\
&= N_0\frac{E}{2}\mathrm{Re}\int_{-1}^{1}dt\frac{1}{\sqrt{|\Delta_0|^2 t^2 + E^2 - |\Delta_0|^2}} && (30)\\
&= N_0\frac{E}{2|\Delta_0|}\mathrm{Re}\int_{-1}^{1}dt\frac{1}{\sqrt{t^2 + \frac{E^2 - |\Delta_0|^2}{|\Delta_0|^2}}} && (31)\\
&= N_0\frac{E}{|\Delta_0|}\mathrm{Re}\int_0^{1}dt\frac{1}{\sqrt{t^2 + \frac{E^2 - |\Delta_0|^2}{|\Delta_0|^2}}} && (32)\\
&= N_0\frac{E}{|\Delta_0|}\mathrm{Re}\int_0^{1}dt\frac{1}{\sqrt{t^2 + a}}, \qquad \left(a \equiv \frac{E^2 - |\Delta_0|^2}{|\Delta_0|^2}\right) && (33)
\end{aligned}$$

$$= N_0 \frac{E}{|\Delta_0|} \mathrm{Re} \int_0^1 dt \frac{t+\sqrt{t^2+a}}{\sqrt{t^2+a}\left(t+\sqrt{t^2+a}\right)} \tag{34}$$

$$= N_0 \frac{E}{|\Delta_0|} \mathrm{Re} \int_0^1 dt \frac{1+\frac{2t}{2\sqrt{t^2+a}}}{t+\sqrt{t^2+a}} \tag{35}$$

$$= N_0 \frac{E}{|\Delta_0|} \mathrm{Re} \left[\ln \left| \left(t+\sqrt{t^2+a}\right) \right| \right]_0^1 \tag{36}$$

$$= N_0 \frac{E}{|\Delta_0|} \mathrm{Re} \left[\ln \left| 1+\sqrt{1+a} \right| - \ln \left| \sqrt{a} \right| \right] \tag{37}$$

$$= N_0 \frac{E}{|\Delta_0|} \left[\ln \left| 1+\sqrt{1+a} \right| - \ln \left| \sqrt{a} \right| \right] \tag{38}$$

$$= N_0 \frac{E}{|\Delta_0|} \ln \left| \sqrt{\frac{1}{a}} + \sqrt{1+\frac{1}{a}} \right| \tag{39}$$

$$= N_0 \frac{E}{|\Delta_0|} \ln \left| \sqrt{\frac{|\Delta_0|^2}{E^2-|\Delta_0|^2}} + \sqrt{1+\frac{|\Delta_0|^2}{E^2-|\Delta_0|^2}} \right| \tag{40}$$

$$= N_0 \frac{E}{|\Delta_0|} \ln \left| \sqrt{\frac{|\Delta_0|^2}{E^2-|\Delta_0|^2}} + \sqrt{\frac{E^2}{E^2-|\Delta_0|^2}} \right| \tag{41}$$

$$= N_0 \frac{E}{|\Delta_0|} \ln \left| \frac{|\Delta_0|}{\sqrt{E^2-|\Delta_0|^2}} + \frac{E}{\sqrt{E^2-|\Delta_0|^2}} \right|, \qquad (E \geq 0) \tag{42}$$

$$= N_0 \frac{E}{|\Delta_0|} \ln \left| \frac{E+|\Delta_0|}{\sqrt{E^2-|\Delta_0|^2}} \right| \tag{43}$$

$$= N_0 \frac{E}{|\Delta_0|} \ln \left| \sqrt{\frac{(E+|\Delta_0|)^2}{(E+|\Delta_0|)(E-|\Delta_0|)}} \right| \tag{44}$$

$$= N_0 \frac{E}{|\Delta_0|} \ln \left| \sqrt{\frac{E+|\Delta_0|}{E-|\Delta_0|}} \right| \tag{45}$$

$$= N_0 \frac{E}{2|\Delta_0|} \ln \left| \frac{E+|\Delta_0|}{E-|\Delta_0|} \right|. \tag{46}$$

Here, the *"Mathematica"* has been utilized for the integration actually.

For $0 \leq E \ll |\Delta_0|$,

$$\ln \left| \frac{E+|\Delta_0|}{E-|\Delta_0|} \right| = \ln \left| \frac{1+E/|\Delta_0|}{1-E/|\Delta_0|} \right| \tag{47}$$

$$= \ln \left| 1+E/|\Delta_0| \right| - \ln \left| 1-E/|\Delta_0| \right| \tag{48}$$

$$\approx +E/|\Delta_0| - (-E/|\Delta_0|) \tag{49}$$

$$= \frac{2E}{|\Delta_0|}, \tag{50}$$

and therefore

$$N(E) \approx N_0 \frac{E^2}{|\Delta_0|^2}. \tag{51}$$

This means that when the gap has point nodes, the density of states $N(E) \propto E^2$ for $E \simeq 0$.

(2) For the BW state $\vec{d}_k = \Delta_0(\hat{k}_x, \hat{k}_y, \hat{k}_z)$ [which is a full-gap state without gap nodes],

$$\begin{aligned} |\Delta_k|^2 &= |d_k^x|^2 + |d_k^y|^2 + |d_k^z|^2 && (52)\\ &= |\Delta_0|^2(\hat{k}_x^2 + \hat{k}_y^2 + \hat{k}_z^2) && (53)\\ &= |\Delta_0|^2\Big((\cos\phi\sin\theta)^2 + (\sin\phi\sin\theta)^2 + (\cos\theta)^2\Big) && (54)\\ &= |\Delta_0|^2(\sin^2\theta + \cos^2\theta) && (55)\\ &= |\Delta_0|^2. && (56) \end{aligned}$$

Inserting this into the expression for $N(E)$ (for $E \geq 0$) in Eq. (11),

$$\begin{aligned} N(E) &= N_0 \int \frac{d\Omega_k}{4\pi} \mathrm{Re}\frac{E}{\sqrt{E^2 - |\Delta_k|^2}} && (57)\\ &= N_0 \int \frac{d\Omega_k}{4\pi} \mathrm{Re}\frac{E}{\sqrt{E^2 - |\Delta_0|^2}} && (58)\\ &= N_0 \mathrm{Re}\left\{\frac{E}{\sqrt{E^2 - |\Delta_0|^2}}\right\} \int \frac{d\Omega_k}{4\pi} && (59)\\ &= N_0 \mathrm{Re}\left\{\frac{E}{\sqrt{E^2 - |\Delta_0|^2}}\right\} && (60)\\ &= \begin{cases} N_0 \mathrm{Re}\left\{\dfrac{E}{\sqrt{E^2 - |\Delta_0|^2}}\right\} & (E > |\Delta_0|) \\ N_0 \mathrm{Re}\left\{\dfrac{E}{i\sqrt{|\Delta_0|^2 - E^2}}\right\} & (0 \leq E < |\Delta_0|) \end{cases} && (61)\\ &= \begin{cases} N_0 \dfrac{E}{\sqrt{E^2 - |\Delta_0|^2}} & (E > |\Delta_0|) \\ 0 & (0 \leq E < |\Delta_0|) \end{cases} && (62) \end{aligned}$$

(3) For the polar state $\vec{d}_k = \Delta_0(0, 0, \hat{k}_z)$ [which has a line gap node along $\hat{k}_z = 0$ (i.e., along the line $\theta = \pi/2$)],

$$\begin{aligned} |\Delta_k|^2 &= |d_k^x|^2 + |d_k^y|^2 + |d_k^z|^2 && (63)\\ &= |\Delta_0|^2 \hat{k}_z^2 && (64)\\ &= |\Delta_0|^2 \cos^2\theta. && (65) \end{aligned}$$

Inserting this into the expression for $N(E)$ (for $E \geq 0$) in Eq. (11),

$$\begin{aligned} N(E) &= N_0 \int \frac{d\Omega_k}{4\pi} \mathrm{Re}\frac{E}{\sqrt{E^2 - |\Delta_k|^2}} && (66)\\ &= N_0 \int \frac{d\Omega_k}{4\pi} \mathrm{Re}\frac{E}{\sqrt{E^2 - |\Delta_0|^2 \hat{k}_z^2}} && (67) \end{aligned}$$

$$
\begin{aligned}
&= N_0 \int \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \mathrm{Re} \frac{E}{\sqrt{E^2 - |\Delta_0|^2 \cos^2\theta}} && (68)\\
&= N_0 \frac{E}{2} \int_0^{\pi} d\theta \sin\theta \mathrm{Re} \frac{1}{\sqrt{E^2 - |\Delta_0|^2 \cos^2\theta}} && (69)\\
&= N_0 \frac{E}{2} \int_{-1}^{1} dt \mathrm{Re} \frac{1}{\sqrt{E^2 - |\Delta_0|^2 t^2}}, \qquad (t = \cos\theta) && (70)\\
&= N_0 E \int_0^{1} dt \mathrm{Re} \frac{1}{\sqrt{E^2 - |\Delta_0|^2 t^2}} && (71)
\end{aligned}
$$

When $E > |\Delta_0|$,

$$
\begin{aligned}
N(E) &= N_0 E \int_0^{1} dt \mathrm{Re} \frac{1}{\sqrt{E^2 - |\Delta_0|^2 t^2}} && (72)\\
&= N_0 E \int_0^{1} dt \frac{1}{\sqrt{E^2 - |\Delta_0|^2 t^2}} && (73)\\
&= N_0 \frac{E}{|\Delta_0|} \int_0^{1} dt \frac{1}{\sqrt{a^2 - t^2}}, \qquad \Big(a \equiv \frac{E}{|\Delta_0|} > 1\Big) && (74)\\
&= N_0 \frac{E}{|\Delta_0|} \left[-\arctan\left(\frac{t\sqrt{a^2 - t^2}}{-(a^2 - t^2)} \right) \right]_{t=0}^{1} \qquad \Big(-\frac{\pi}{2} \le \arctan(\cdots) \le \frac{\pi}{2}\Big) && (75)\\
&= N_0 \frac{E}{|\Delta_0|} \left[\arctan\left(\frac{t\sqrt{a^2 - t^2}}{a^2 - t^2} \right) \right]_{t=0}^{1} && (76)\\
&= N_0 \frac{E}{|\Delta_0|} \arctan\left(\frac{\sqrt{a^2 - 1}}{a^2 - 1} \right) && (77)\\
&= N_0 \frac{E}{|\Delta_0|} \arctan\left(\frac{1}{\sqrt{a^2 - 1}} \right) && (78)\\
&= N_0 \frac{E}{|\Delta_0|} \arcsin\left(\frac{1}{a} \right), \qquad \Big((\sqrt{a^2 - 1}\ )^2 + 1^2 = a^2 \Big) && (79)\\
&= N_0 \frac{E}{|\Delta_0|} \arcsin\left(\frac{|\Delta_0|}{E} \right). && (80)
\end{aligned}
$$

When $0 \le E < |\Delta_0|$,

$$
\begin{aligned}
N(E) &= N_0 E \int_0^{1} dt \mathrm{Re} \frac{1}{\sqrt{E^2 - |\Delta_0|^2 t^2}} && (81)\\
&= N_0 E \int_0^{t_0 - 0^+} dt \mathrm{Re} \frac{1}{\sqrt{E^2 - |\Delta_0|^2 t^2}} + N_0 E \int_{t_0 + 0^+}^{1} dt \mathrm{Re} \frac{1}{\sqrt{E^2 - |\Delta_0|^2 t^2}}, && (82)\\
&\qquad \Big(t_0 \equiv \frac{E}{|\Delta_0|} \Big) \\
&= N_0 E \int_0^{t_0 - 0^+} dt \mathrm{Re} \frac{1}{\sqrt{E^2 - |\Delta_0|^2 t^2}} + N_0 E \int_{t_0 + 0^+}^{1} dt \mathrm{Re} \frac{1}{i\sqrt{|\Delta_0|^2 t^2 - E^2}} && (83)\\
&= N_0 E \int_0^{t_0 - 0^+} dt \frac{1}{\sqrt{E^2 - |\Delta_0|^2 t^2}} + 0 && (84)
\end{aligned}
$$

$$
\begin{aligned}
&= N_0 \frac{E}{|\Delta_0|} \int_0^{t_0 - 0^+} dt \frac{1}{\sqrt{t_0^2 - t^2}} && (85) \\
&= N_0 \frac{E}{|\Delta_0|} \left[\arctan\left(\frac{t\sqrt{t_0^2 - t^2}}{t_0^2 - t^2} \right) \right]_{t=0}^{t_0 - 0^+} \qquad \left(-\frac{\pi}{2} \leq \arctan(\cdots) \leq \frac{\pi}{2} \right) && (86) \\
&= N_0 \frac{E}{|\Delta_0|} \left[\arctan\left(\frac{t}{\sqrt{t_0^2 - t^2}} \right) \right]_{t=0}^{t_0 - 0^+} && (87) \\
&= N_0 \frac{E}{|\Delta_0|} \left[\arctan\left(+\infty \right) - 0 \right] && (88) \\
&= N_0 \frac{E}{|\Delta_0|} \frac{\pi}{2}. && (89)
\end{aligned}
$$

Hence,

$$
N(E) = \begin{cases} N_0 \dfrac{E}{|\Delta_0|} \arcsin\left(\dfrac{|\Delta_0|}{E} \right) & (E > |\Delta_0|) \\ N_0 \dfrac{E}{|\Delta_0|} \dfrac{\pi}{2} & (0 \leq E < |\Delta_0|) \end{cases} \tag{90}
$$

This result indicates that when the gap has a line node, the density of states $N(E) \propto E$ for $E \simeq 0$.