Unkonventionelle Supraleitung WS 05/06
Lösungen zur Serie 7

7.1 The equation of motion for the oscillating wire is

$$(\mu+\mu_L)\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2} - D\frac{\partial y}{\partial t} - I_0 B e^{i\omega t}. \tag{1}$$

a) First, we assume $I_0 = 0$ and

$$y(x,t) = \sin\Big(\frac{n\pi x}{l}\Big)e^{i\tilde{\omega}t}. \tag{2}$$

In this case, from the equation of motion,

$$-(\mu+\mu_L)\tilde{\omega}^2 = -T\Big(\frac{n\pi}{l}\Big)^2 - iD\tilde{\omega}, \tag{3}$$

$$\rightarrow \quad (\mu+\mu_L)\tilde{\omega}^2 - iD\tilde{\omega} - T\Big(\frac{n\pi}{l}\Big)^2 = 0, \tag{4}$$

$$\rightarrow \quad \tilde{\omega} = \frac{1}{2(\mu+\mu_L)}\left(iD \pm \sqrt{-D^2 + 4(\mu+\mu_L)T\Big(\frac{n\pi}{l}\Big)^2}\right). \tag{5}$$

Thus, $\tilde{\omega}$ is expressed as

$$\tilde{\omega} = i\alpha \pm \omega_n, \tag{6}$$

where

$$\alpha = \frac{D}{2(\mu+\mu_L)}, \tag{7}$$

and

$$\omega_n = \sqrt{\frac{T}{\mu+\mu_L}\Big(\frac{n\pi}{l}\Big)^2 - \alpha^2}. \tag{8}$$

b) Next, we assume $I_0 \neq 0$. Let us consider a solution of Eq. (1) with the form

$$y(x,t) = \sum_n A_n \sin\Big(\frac{n\pi x}{l}\Big)e^{i\tilde{\omega}t}, \tag{9}$$

where n is the positive integer and we assume that A_n is independent of the time t. Inserting this into Eq. (1),

$$\begin{aligned}(\mu+\mu_L)(i\tilde{\omega})^2\sum_n A_n \sin\Big(\frac{n\pi x}{l}\Big)e^{i\tilde{\omega}t} &= -T\Big(\frac{n\pi}{l}\Big)^2\sum_n A_n \sin\Big(\frac{n\pi x}{l}\Big)e^{i\tilde{\omega}t} \\ &\quad - Di\tilde{\omega}\sum_n A_n \sin\Big(\frac{n\pi x}{l}\Big)e^{i\tilde{\omega}t} \\ &\quad - I_0 B e^{i\omega t}.\end{aligned} \tag{10}$$

Using the orthogonality relations

$$\int_0^l dx \sin\Big(\frac{m\pi x}{l}\Big)\sin\Big(\frac{n\pi x}{l}\Big) = \frac{l}{2}\delta_{m,n}, \tag{11}$$

from Eq. (10) we obtain

$$\begin{aligned}\frac{l}{2}(\mu+\mu_L)(i\tilde{\omega})^2 A_n e^{i\tilde{\omega}t} &= -\frac{l}{2}T\Big(\frac{n\pi}{l}\Big)^2 A_n e^{i\tilde{\omega}t} \\ &\quad -\frac{l}{2}Di\tilde{\omega}A_n e^{i\tilde{\omega}t} \\ &\quad -I_0 B e^{i\omega t}\int_0^l dx\sin\Big(\frac{n\pi x}{l}\Big). \end{aligned} \tag{12}$$

Because

$$\int_0^l dx\sin\Big(\frac{n\pi x}{l}\Big) = \begin{cases} \dfrac{2l}{n\pi} & (n=1,\ 3,\ 5,\ \cdots) \\ 0 & (n=2,\ 4,\ 6,\ \cdots) \end{cases} \tag{13}$$

$$= \frac{2l}{n\pi}\delta_{n,2j-1}, \qquad (j\text{: the positive integer}) \tag{14}$$

Eq. (12) is

$$\begin{aligned}\frac{l}{2}(\mu+\mu_L)(i\tilde{\omega})^2 A_n e^{i\tilde{\omega}t} &= -\frac{l}{2}T\Big(\frac{n\pi}{l}\Big)^2 A_n e^{i\tilde{\omega}t} \\ &\quad -\frac{l}{2}Di\tilde{\omega}A_n e^{i\tilde{\omega}t} \\ &\quad -I_0 B e^{i\omega t}\frac{2l}{n\pi}\delta_{n,2j-1}. \end{aligned} \tag{15}$$

From this,

$$\Big[(\mu+\mu_L)(i\tilde{\omega})^2 + T\Big(\frac{n\pi}{l}\Big)^2 + Di\tilde{\omega}\Big]e^{i(\tilde{\omega}-\omega)t}A_n = -I_0 B\frac{4}{n\pi}\delta_{n,2j-1}. \tag{16}$$

Thus,

$$A_n = \delta_{n,2j-1}\Big(\frac{4I_0B}{n\pi}\Big)\frac{e^{-i(\tilde{\omega}-\omega)t}}{(\mu+\mu_L)\tilde{\omega}^2 - T\Big(\frac{n\pi}{l}\Big)^2 - iD\tilde{\omega}}. \tag{17}$$

Because of the assumption that A_n is independent of t,

$$\tilde{\omega} = \omega, \tag{18}$$

and then

$$\begin{aligned} A_n &= \delta_{n,2j-1}\Big(\frac{4I_0B}{n\pi}\Big)\frac{1}{(\mu+\mu_L)\omega^2 - T\Big(\frac{n\pi}{l}\Big)^2 - iD\omega} && (19)\\ &= \delta_{n,2j-1}\Big(\frac{4I_0B}{n\pi\mu}\Big)\frac{1}{\Big(1+\frac{\mu_L}{\mu}\Big)\omega^2 - \frac{T}{\mu}\Big(\frac{n\pi}{l}\Big)^2 - i\frac{D}{\mu}\omega} && (20)\\ &= \delta_{n,2j-1}\Big(\frac{4I_0B}{n\pi\mu}\Big)\frac{\Big\{\Big(1+\frac{\mu_L}{\mu}\Big)\omega^2 - \frac{T}{\mu}\Big(\frac{n\pi}{l}\Big)^2\Big\} + i\frac{D}{\mu}\omega}{\Big\{\Big(1+\frac{\mu_L}{\mu}\Big)\omega^2 - \frac{T}{\mu}\Big(\frac{n\pi}{l}\Big)^2\Big\}^2 + \Big(\frac{D}{\mu}\omega\Big)^2}. && (21)\end{aligned}$$

c) In the present case, the velocity is $\mathbf{v}(t) = \Big(0, \frac{\partial y(x,t)}{\partial t}, 0\Big)$. When $\mathbf{B} = (0, 0, B)$,

$$
\begin{aligned}
V(t) &= \int_0^L (\mathbf{v}(t) \times \mathbf{B}) \cdot d\mathbf{l} && (22)\\
&= \int_0^L \Big(\frac{\partial y(x,t)}{\partial t} B, 0, 0\Big) \cdot d\mathbf{l} && (23)\\
&= B \int_0^l \frac{\partial y(x,t)}{\partial t} dx. && (24)
\end{aligned}
$$

From Eq. (9) with $\tilde{\omega} = \omega$,

$$
\begin{aligned}
\frac{\partial y(x,t)}{\partial t} &= \frac{\partial}{\partial t}\Big[\sum_n A_n \sin\Big(\frac{n\pi x}{l}\Big) e^{i\omega t}\Big] && (25)\\
&= i\omega \sum_n A_n \sin\Big(\frac{n\pi x}{l}\Big) e^{i\omega t}. && (26)
\end{aligned}
$$

Then,

$$
\begin{aligned}
V(t) &= B \int_0^l \frac{\partial y(x,t)}{\partial t} dx && (27)\\
&= B \int_0^l i\omega \sum_n A_n \sin\Big(\frac{n\pi x}{l}\Big) e^{i\omega t} dx && (28)\\
&= Bi\omega \sum_n A_n e^{i\omega t} \int_0^l \sin\Big(\frac{n\pi x}{l}\Big) dx && (29)\\
&= Bi\omega \sum_n A_n e^{i\omega t} \frac{2l}{n\pi} \delta_{n,2j-1} && (30)\\
&= \sum_n \delta_{n,2j-1} \frac{2lB}{n\pi} i\omega e^{i\omega t} A_n && (31)\\
&= \sum_n \delta_{n,2j-1} \frac{2lB}{n\pi} i\omega e^{i\omega t} \Big(\frac{4I_0 B}{n\pi\mu}\Big) \frac{1}{\Big(1 + \frac{\mu_L}{\mu}\Big)\omega^2 - \frac{T}{\mu}\Big(\frac{n\pi}{l}\Big)^2 - i\frac{D}{\mu}\omega} && (32)\\
&= \sum_n \delta_{n,2j-1} \Big(\frac{8I_0 B^2 l}{n^2\pi^2\mu}\Big) \frac{i\omega e^{i\omega t}}{\Big(1 + \frac{\mu_L}{\mu}\Big)\omega^2 - \frac{T}{\mu}\Big(\frac{n\pi}{l}\Big)^2 - i\frac{D}{\mu}\omega} && (33)\\
&= \sum_{j=1}^{\infty} \Big(\frac{8I_0 B^2 l}{(2j-1)^2\pi^2\mu}\Big) \frac{i\omega e^{i\omega t}}{\Big(1 + \frac{\mu_L}{\mu}\Big)\omega^2 - \frac{T}{\mu}\Big(\frac{(2j-1)\pi}{l}\Big)^2 - i\frac{D}{\mu}\omega}. && (34)
\end{aligned}
$$

7.2 Let us consider the following implicit equation for the depression of the critical temperature T_c in the presence of a pair breaking mechanism characterized by τ:

$$\ln\Big(\frac{1}{x}\Big) = \Psi\Big(\frac{1}{2} + \frac{y}{4\pi x}\Big) - \Psi\Big(\frac{1}{2}\Big). \tag{35}$$

Here, $x = T_{\mathrm{c}}/T_{\mathrm{c0}}$, $y = 2\hbar/\tau k_{\mathrm{B}} T_{\mathrm{c0}}$, and T_{c0} is the original critical temperature in the absence of pair breaking effects.

The digamma function Ψ can be approximated as

$$\Psi\Big(\frac{1}{2} + z\Big) \approx \Psi\Big(\frac{1}{2}\Big) + \frac{\pi^2}{2} z \qquad (z \to 0), \tag{36}$$

$$\Psi(z) \approx \ln z - \frac{1}{2z} - \frac{1}{12z^2} \qquad (z \to \infty), \tag{37}$$

and

$$\Psi(\frac{1}{2}) = -\ln(4 \times e^{\gamma}) \approx -\ln(4 \times 1.78) \approx -1.96. \quad (\gamma \approx 0.5772) \tag{38}$$

a) For $x \to 0$,

$$\Psi\Big(\frac{1}{2} + \frac{y}{4\pi x}\Big) \approx \Psi\Big(\frac{y}{4\pi x}\Big) \tag{39}$$

$$\approx \ln\Big(\frac{y}{4\pi x}\Big). \tag{40}$$

where we have used Eq. (37). Inserting this into Eq. (35),

$$\ln\Big(\frac{1}{x}\Big) = \ln\Big(\frac{y}{4\pi x}\Big) - \Psi\Big(\frac{1}{2}\Big), \tag{41}$$

$$\to \quad \ln\Big(\frac{y}{4\pi}\Big) = \Psi\Big(\frac{1}{2}\Big), \tag{42}$$

$$\to \quad \ln\Big(\frac{y}{4\pi}\Big) \approx -\ln(4 \times 1.78), \tag{43}$$

$$\to \quad \frac{y}{4\pi} = e^{-\ln(4\times 1.78)}, \tag{44}$$

$$\to \quad \frac{y}{4\pi} = \frac{1}{e^{\ln(4\times 1.78)}}, \tag{45}$$

$$\to \quad \frac{y}{4\pi} = \frac{1}{4 \times 1.78}, \tag{46}$$

$$\to \quad y = \frac{\pi}{1.78} \tag{47}$$

$$\approx 1.76, \tag{48}$$

where we have used Eq. (38).

Because $y = 2\hbar/\tau k_{\mathrm{B}} T_{\mathrm{c0}}$,

$$\frac{2\hbar}{\tau k_{\mathrm{B}} T_{\mathrm{c0}}} \approx 1.76, \tag{49}$$

$$\to \quad \frac{\hbar}{\tau} \approx \frac{1.76}{2} k_{\mathrm{B}} T_{\mathrm{c0}}, \tag{50}$$

Thus,

$$\frac{\hbar}{\tau} \sim k_{\mathrm{B}} T_{\mathrm{c0}} \qquad (T_{\mathrm{c}} \to 0). \tag{51}$$

b) For $x \to 1$, from Eq. (35),

$$0 = \Psi\Big(\frac{1}{2} + \frac{y}{4\pi}\Big) - \Psi\Big(\frac{1}{2}\Big). \tag{52}$$

This means that $y \to 0$ for $x \to 1$. Therefore, using Eq. (36),

$$\Psi\Big(\frac{1}{2} + \frac{y}{4\pi x}\Big) \approx \Psi\Big(\frac{1}{2}\Big) + \frac{\pi^2}{2}\frac{y}{4\pi x}. \tag{53}$$

Inserting this into Eq. (35),

$$\begin{aligned}
& \ln\Big(\frac{1}{x}\Big) &=& \ \Psi\Big(\frac{1}{2}\Big) + \frac{\pi^2}{2}\frac{y}{4\pi x} - \Psi\Big(\frac{1}{2}\Big), & (54)\\
\to \quad & \ln\Big(\frac{1}{x}\Big) &=& \ \frac{\pi^2}{2}\frac{y}{4\pi x}, & (55)\\
\to \quad & y &=& -\left(\frac{8}{\pi}\right) x \ln x, & (56)\\
\to \quad & y &=& -\left(\frac{8}{\pi}\right)\{1-(1-x)\}\cdot \ln\big[1-(1-x)\big] & (57)\\
& &\approx& -\left(\frac{8}{\pi}\right)\{1-(1-x)\}\cdot \big[-(1-x)\big] & (58)\\
& &=& \left(\frac{8}{\pi}\right)\{1-(1-x)\}\cdot (1-x) & (59)\\
& &=& \left(\frac{8}{\pi}\right)\{(1-x)-(1-x)^2\} & (60)\\
& &\approx& \left(\frac{8}{\pi}\right)(1-x). & (61)
\end{aligned}$$

Because $y = 2\hbar/\tau k_{\rm B} T_{\rm c0}$ and $x = T_{\rm c}/T_{\rm c0}$,

$$\begin{aligned}
& \frac{2\hbar}{\tau k_{\rm B} T_{\rm c0}} &\approx& \left(\frac{8}{\pi}\right)\Big(1 - \frac{T_{\rm c}}{T_{\rm c0}}\Big), & (62)\\
\to \quad & \frac{\hbar}{\tau} &\approx& \left(\frac{4}{\pi}\right) k_{\rm B} T_{\rm c0}\Big(1 - \frac{T_{\rm c}}{T_{\rm c0}}\Big), & (63)
\end{aligned}$$

Thus,

$$\frac{\hbar}{\tau} \sim k_{\rm B}\Big(T_{\rm c0} - T_{\rm c}\Big) \qquad (T_{\rm c} \to T_{\rm c0}). \tag{64}$$