Unkonventionelle Supraleitung

Serie 8

Verteilung: 20.Dezember Abgabe: 10.Januar

The generalized BCS theory with the Zeeman effect.

8.1 Let us consider the following Hamiltonian for a superconductor under a magnetic field,

$$H = H_{\mathrm{BCS}} + H_{\mathrm{Z}}.$$

H_{BCS} is the same as the Hamiltonian in the problem **4.1** in Serie 4 (the notations are also the same):

$$H_{\mathrm{BCS}} = \sum_k C_k^\dagger \check{\varepsilon}_k C_k,$$

$$C_k^\dagger = \left(c_{k\uparrow}^\dagger, \quad c_{k\downarrow}^\dagger, \quad c_{-k\uparrow}, \quad c_{-k\downarrow} \right), \qquad \check{\varepsilon}_k = \frac{1}{2}\begin{pmatrix} \xi_k\hat{\sigma}_0 & \hat{\Delta}_k \\ \hat{\Delta}_k^\dagger & -\xi_k\hat{\sigma}_0 \end{pmatrix}, \qquad C_k = \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \\ c_{-k\uparrow}^\dagger \\ c_{-k\downarrow}^\dagger \end{pmatrix}.$$

The Zeeman term H_{Z} is written with the Pauli matrices $\hat{\vec{\sigma}}$ and the magnetic field $\vec{H}$ as

$$\begin{aligned} H_{\mathrm{Z}} &= -\mu_{\mathrm{B}} \sum_{k,s_1,s_2} c_{ks_1}^\dagger (\vec{\sigma}_{s_1 s_2}\cdot\vec{H}) c_{ks_2} \\ &= \sum_k C_k^\dagger \frac{1}{2}\begin{pmatrix} -\mu_{\mathrm{B}}(\hat{\vec{\sigma}}\cdot\vec{H}) & 0 \\ 0 & \mu_{\mathrm{B}}(\hat{\vec{\sigma}}^T\cdot\vec{H}) \end{pmatrix} C_k, \end{aligned}$$

where we have omitted the c-number term in the last line.

Hence,

$$\begin{aligned} H &= H_{\mathrm{BCS}} + H_{\mathrm{Z}} \\ &= \sum_k C_k^\dagger \frac{1}{2}\begin{pmatrix} \xi_k\hat{\sigma}_0 & \hat{\Delta}_k \\ \hat{\Delta}_k^\dagger & -\xi_k\hat{\sigma}_0 \end{pmatrix} C_k + \sum_k C_k^\dagger \frac{1}{2}\begin{pmatrix} -\mu_{\mathrm{B}}(\hat{\vec{\sigma}}\cdot\vec{H}) & 0 \\ 0 & \mu_{\mathrm{B}}(\hat{\vec{\sigma}}^T\cdot\vec{H}) \end{pmatrix} C_k \\ &= \sum_k C_k^\dagger \frac{1}{2}\begin{pmatrix} \xi_k\hat{\sigma}_0 - \mu_{\mathrm{B}}(\hat{\vec{\sigma}}\cdot\vec{H}) & \hat{\Delta}_k \\ \hat{\Delta}_k^\dagger & -\left(\xi_k\hat{\sigma}_0 - \mu_{\mathrm{B}}(\hat{\vec{\sigma}}^T\cdot\vec{H})\right) \end{pmatrix} C_k \\ &\equiv \sum_k C_k^\dagger \frac{1}{2}\begin{pmatrix} \hat{T}_{1k} & \hat{\Delta}_k \\ \hat{\Delta}_k^\dagger & -\hat{T}_{2k} \end{pmatrix} C_k \\ &\equiv \sum_k C_k^\dagger \check{\varepsilon}'_k C_k, \end{aligned}$$

where

$$\begin{aligned} \hat{T}_{1k} &= \xi_k\hat{\sigma}_0 - \mu_{\mathrm{B}}(\hat{\vec{\sigma}}\cdot\vec{H}), \\ \hat{T}_{2k} &= \xi_k\hat{\sigma}_0 - \mu_{\mathrm{B}}(\hat{\vec{\sigma}}^T\cdot\vec{H}). \end{aligned}$$

Here, $\hat{T}_{1,-k} = \hat{T}_{1k}$ and $\hat{T}_{2,-k} = \hat{T}_{2k}$ because $\xi_{-k} = \xi_k$. One can also confirm easily that $\hat{T}_{2k}^* = \hat{T}_{1k}$ because ξ_k and $\vec{H}$ are real. From now on, let us assume that $\vec{H} = (0, 0, H_z) \parallel \hat{z}$ and $\hat{u}_k = u_k^0\hat{\sigma}_0 + u_k^z\hat{\sigma}_z$. When $\vec{H} \parallel \hat{z}$, $\hat{T}_{1k} = \hat{T}_{2k}$.

The Hamiltonian H is diagonalized as

$$\begin{aligned} H &= \sum_k \left(C_k^\dagger \check{U}_k\right)\left(\check{U}_k^\dagger \check{\varepsilon}'_k \check{U}_k\right)\left(\check{U}_k^\dagger C_k\right) \\ &= \sum_k A_k^\dagger \check{E}_k A_k, \end{aligned}$$

with

$$\check{E}_k = \frac{1}{2}\begin{pmatrix} \hat{E}_k & 0 \\ 0 & -\hat{E}_{-k} \end{pmatrix} \qquad \text{and} \qquad \hat{E}_k = \begin{pmatrix} E_{k,+} & 0 \\ 0 & E_{k,-} \end{pmatrix},$$

by the Bogoliubov transformation:

$$A_k = \check{U}_k^\dagger C_k, \qquad \check{U}_k = \begin{pmatrix} \hat{u}_k & \hat{v}_k \\ \hat{v}_{-k}^* & \hat{u}_{-k}^* \end{pmatrix}, \qquad \check{U}_k^\dagger = \begin{pmatrix} \hat{u}_k^\dagger & \hat{v}_{-k}^T \\ \hat{v}_k^\dagger & \hat{u}_{-k}^T \end{pmatrix}.$$

Here, $\check{U}_k^\dagger \check{U}_k = \check{U}_k \check{U}_k^\dagger = \check{1}$.

a) Show that in the case of the singlet state $\hat{\Delta}_k = \Psi i\hat{\sigma}_y$, the eigen values are given as $E_{k,+} = \sqrt{\xi_k^2 + |\Delta_k|^2} - \mu_{\mathrm{B}} H_z$, and $E_{k,-} = \sqrt{\xi_k^2 + |\Delta_k|^2} + \mu_{\mathrm{B}} H_z$.

b) Show that in the case of the unitary triplet state $\hat{\Delta}_k = \vec{d}_k \cdot \hat{\vec{\sigma}} i\hat{\sigma}_y$ with $\vec{d} \perp \vec{H}$, namely with $\vec{d} = (d_x, d_y, 0)$, the eigen values are given as
$E_{k,+} = \sqrt{(\xi_k - \mu_{\mathrm{B}} H_z)^2 + |\Delta_k|^2}$, and $E_{k,-} = \sqrt{(\xi_k + \mu_{\mathrm{B}} H_z)^2 + |\Delta_k|^2}$.

c) Show that in the case of the unitary triplet state $\hat{\Delta}_k = \vec{d}_k \cdot \hat{\vec{\sigma}} i\hat{\sigma}_y$ with $\vec{d} \parallel \vec{H}$, namely with $\vec{d} = (0, 0, d_z)$, the eigen values are given as
$E_{k,+} = \sqrt{\xi_k^2 + |\Delta_k|^2} - \mu_{\mathrm{B}} H_z$, and $E_{k,-} = \sqrt{\xi_k^2 + |\Delta_k|^2} + \mu_{\mathrm{B}} H_z$.

<u>Hint</u>: Considering the equation $\check{\varepsilon}'_k \check{U}_k = \check{U}_k \check{E}_k$, four equations will be obtained. Only two of them are independent equations:

$$\begin{aligned} \hat{T}_{1k}\hat{u}_k + \hat{\Delta}_k \hat{v}_{-k}^* &= \hat{u}_k \hat{E}_k, \\ \hat{\Delta}_k^\dagger \hat{u}_k - \hat{T}_{2k}\hat{v}_{-k}^* &= \hat{v}_{-k}^* \hat{E}_k. \end{aligned}$$

From the former equation, one can calculate $\hat{v}_{-k}^*$ using the property of the singlet and unitary triplet states: $\hat{\Delta}_k \hat{\Delta}_k^\dagger = \hat{\Delta}_k^\dagger \hat{\Delta}_k = |\Delta_k|^2 \hat{\sigma}_0$ with $|\Delta_k|^2 \equiv \frac{1}{2}\mathrm{Tr}[\hat{\Delta}_k \hat{\Delta}_k^\dagger]$. Substituting $\hat{v}_{-k}^*$ into the latter equation, one will obtain the following equation.

$$|\Delta_k|^4 \hat{u}_k - \hat{\Delta}_k \hat{T}_{2k} \hat{\Delta}_k^\dagger \left(\hat{u}_k \hat{E}_k - \hat{T}_{1k}\hat{u}_k\right) = |\Delta_k|^2 \left(\hat{u}_k \hat{E}_k - \hat{T}_{1k}\hat{u}_k\right) \hat{E}_k.$$

Assume that $\vec{H} = (0, 0, H_z) \parallel \hat{z}$ and $\hat{u}_k = u_k^0 \hat{\sigma}_0 + u_k^z \hat{\sigma}_z$. Then, for each case of the pairing states, calculating each term by explicitly considering the matrix elements, one will obtain equations for $E_{k,\pm}$.

8.2 Let us consider the same situation as in the problem **8.1**.

Calculate $\hat{u}_k$ and $\hat{v}_k$ for the singlet state, the unitary triplet state with $\vec{d} \perp \vec{H}$, and the unitary triplet state with $\vec{d} \parallel \vec{H}$.

Hint:

As mentioned above, one can express $\hat{v}^*_{-k}$ by $\hat{u}_k$ from the equation:

$$\hat{T}_{1k}\hat{u}_k + \hat{\Delta}_k\hat{v}^*_{-k} = \hat{u}_k\hat{E}_k.$$

Then, $\hat{v}_k$ and $\hat{v}^\dagger_k$ are obtained from $\hat{v}^*_{-k}$.

Owing to $\check{U}_k\check{U}^\dagger_k = \check{1}$, two independent equations are obtained:

$$\begin{aligned} \hat{u}_k\hat{u}^\dagger_k + \hat{v}_k\hat{v}^\dagger_k &= \hat{\sigma}_0, \\ \hat{v}^*_{-k}\hat{u}^\dagger_k + \hat{u}^*_{-k}\hat{v}^\dagger_k &= 0. \end{aligned}$$

From these equations, one can determine $\hat{u}_k$ and $\hat{v}_k$ for each pairing state.