Unkonventionelle Supraleitung WS 05/06
Lösungen zur Serie 8

The generalized BCS theory with the Zeeman effect.

8.1 Let us consider the following Hamiltonian for a superconductor under a magnetic field,

$$H = H_{\mathrm{BCS}} + H_{\mathrm{Z}}. \tag{1}$$

H_{BCS} is the same as the Hamiltonian in the problem **4.1** in Serie 4 (the notations are also the same):

$$H_{\mathrm{BCS}} = \sum_k C_k^\dagger \check{\varepsilon}_k C_k, \tag{2}$$

$$C_k^\dagger = (\, c_{k\uparrow}^\dagger, \quad c_{k\downarrow}^\dagger, \quad c_{-k\uparrow}, \quad c_{-k\downarrow}\,), \qquad \check{\varepsilon}_k = \frac{1}{2}\begin{pmatrix} \xi_k\hat{\sigma}_0 & \hat{\Delta}_k \\ \hat{\Delta}_k^\dagger & -\xi_k\hat{\sigma}_0 \end{pmatrix}, \qquad C_k = \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \\ c_{-k\uparrow}^\dagger \\ c_{-k\downarrow}^\dagger \end{pmatrix}. \tag{3}$$

The Zeeman term H_{Z} is written with the Pauli matrices $\hat{\vec{\sigma}}$ and the magnetic field $\vec{H}$ as

$$
\begin{aligned}
H_{\mathrm{Z}} &= -\mu_{\mathrm{B}} \sum_{k,s_1,s_2} c_{ks_1}^\dagger (\vec{\sigma}_{s_1 s_2}\cdot\vec{H}) c_{ks_2} && (4)\\
&= -\mu_{\mathrm{B}} \sum_k \Big[(\vec{\sigma}_{\uparrow\uparrow}\cdot\vec{H}) c_{k\uparrow}^\dagger c_{k\uparrow} + (\vec{\sigma}_{\uparrow\downarrow}\cdot\vec{H}) c_{k\uparrow}^\dagger c_{k\downarrow} + (\vec{\sigma}_{\downarrow\uparrow}\cdot\vec{H}) c_{k\downarrow}^\dagger c_{k\uparrow} + (\vec{\sigma}_{\downarrow\downarrow}\cdot\vec{H}) c_{k\downarrow}^\dagger c_{k\downarrow} \Big] && (5)\\
&= -\frac{\mu_{\mathrm{B}}}{2} \sum_k \Big[(\vec{\sigma}_{\uparrow\uparrow}\cdot\vec{H}) c_{k\uparrow}^\dagger c_{k\uparrow} + (\vec{\sigma}_{\uparrow\downarrow}\cdot\vec{H}) c_{k\uparrow}^\dagger c_{k\downarrow} + (\vec{\sigma}_{\downarrow\uparrow}\cdot\vec{H}) c_{k\downarrow}^\dagger c_{k\uparrow} + (\vec{\sigma}_{\downarrow\downarrow}\cdot\vec{H}) c_{k\downarrow}^\dagger c_{k\downarrow} \Big] \\
&\quad - \frac{\mu_{\mathrm{B}}}{2} \sum_k \Big[(\vec{\sigma}_{\uparrow\uparrow}\cdot\vec{H}) c_{-k\uparrow}^\dagger c_{-k\uparrow} + (\vec{\sigma}_{\uparrow\downarrow}\cdot\vec{H}) c_{-k\uparrow}^\dagger c_{-k\downarrow} + (\vec{\sigma}_{\downarrow\uparrow}\cdot\vec{H}) c_{-k\downarrow}^\dagger c_{-k\uparrow} + (\vec{\sigma}_{\downarrow\downarrow}\cdot\vec{H}) c_{-k\downarrow}^\dagger c_{-k\downarrow} \Big] && (6)\\
&= -\frac{\mu_{\mathrm{B}}}{2} \sum_k \Big[(\vec{\sigma}_{\uparrow\uparrow}\cdot\vec{H}) c_{k\uparrow}^\dagger c_{k\uparrow} + (\vec{\sigma}_{\uparrow\downarrow}\cdot\vec{H}) c_{k\uparrow}^\dagger c_{k\downarrow} + (\vec{\sigma}_{\downarrow\uparrow}\cdot\vec{H}) c_{k\downarrow}^\dagger c_{k\uparrow} + (\vec{\sigma}_{\downarrow\downarrow}\cdot\vec{H}) c_{k\downarrow}^\dagger c_{k\downarrow} \Big] \\
&\quad - \frac{\mu_{\mathrm{B}}}{2} \sum_k \Big[(\vec{\sigma}_{\uparrow\uparrow}\cdot\vec{H})(1 - c_{-k\uparrow} c_{-k\uparrow}^\dagger) + (\vec{\sigma}_{\uparrow\downarrow}\cdot\vec{H})(-c_{-k\downarrow} c_{-k\uparrow}^\dagger) \\
&\qquad\qquad + (\vec{\sigma}_{\downarrow\uparrow}\cdot\vec{H})(-c_{-k\uparrow} c_{-k\downarrow}^\dagger) + (\vec{\sigma}_{\downarrow\downarrow}\cdot\vec{H})(1 - c_{-k\downarrow} c_{-k\downarrow}^\dagger) \Big] && (7)\\
&= -\frac{\mu_{\mathrm{B}}}{2} \sum_k \Big[(\vec{\sigma}_{\uparrow\uparrow}\cdot\vec{H}) c_{k\uparrow}^\dagger c_{k\uparrow} + (\vec{\sigma}_{\downarrow\uparrow}\cdot\vec{H}) c_{k\downarrow}^\dagger c_{k\uparrow} + (\vec{\sigma}_{\uparrow\downarrow}\cdot\vec{H}) c_{k\uparrow}^\dagger c_{k\downarrow} + (\vec{\sigma}_{\downarrow\downarrow}\cdot\vec{H}) c_{k\downarrow}^\dagger c_{k\downarrow} \Big] \\
&\quad + \frac{\mu_{\mathrm{B}}}{2} \sum_k \Big[(\vec{\sigma}_{\uparrow\uparrow}\cdot\vec{H}) c_{-k\uparrow} c_{-k\uparrow}^\dagger + (\vec{\sigma}_{\uparrow\downarrow}\cdot\vec{H}) c_{-k\downarrow} c_{-k\uparrow}^\dagger + (\vec{\sigma}_{\downarrow\uparrow}\cdot\vec{H}) c_{-k\uparrow} c_{-k\downarrow}^\dagger + (\vec{\sigma}_{\downarrow\downarrow}\cdot\vec{H}) c_{-k\downarrow} c_{-k\downarrow}^\dagger \Big] \\
&\quad - \frac{\mu_{\mathrm{B}}}{2} \sum_k \Big[(\vec{\sigma}_{\uparrow\uparrow}\cdot\vec{H}) + (\vec{\sigma}_{\downarrow\downarrow}\cdot\vec{H}) \Big] && (8)\\
&= -\frac{\mu_{\mathrm{B}}}{2} \sum_k \big(\, (\vec{\sigma}_{\uparrow\uparrow}\cdot\vec{H}) c_{k\uparrow}^\dagger + (\vec{\sigma}_{\downarrow\uparrow}\cdot\vec{H}) c_{k\downarrow}^\dagger, \quad (\vec{\sigma}_{\uparrow\downarrow}\cdot\vec{H}) c_{k\uparrow}^\dagger + (\vec{\sigma}_{\downarrow\downarrow}\cdot\vec{H}) c_{k\downarrow}^\dagger \,\big) \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \end{pmatrix} \\
&\quad + \frac{\mu_{\mathrm{B}}}{2} \sum_k \big(\, (\vec{\sigma}_{\uparrow\uparrow}\cdot\vec{H}) c_{-k\uparrow} + (\vec{\sigma}_{\uparrow\downarrow}\cdot\vec{H}) c_{-k\downarrow}, \quad (\vec{\sigma}_{\downarrow\uparrow}\cdot\vec{H}) c_{-k\uparrow} + (\vec{\sigma}_{\downarrow\downarrow}\cdot\vec{H}) c_{-k\downarrow} \,\big) \begin{pmatrix} c_{-k\uparrow}^\dagger \\ c_{-k\downarrow}^\dagger \end{pmatrix} \\
&\quad - \frac{\mu_{\mathrm{B}}}{2} \sum_k \Big[(\vec{\sigma}_{\uparrow\uparrow}\cdot\vec{H}) + (\vec{\sigma}_{\downarrow\downarrow}\cdot\vec{H}) \Big] && (9)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\mu_{\rm B}}{2}\sum_k (\, c^\dagger_{k\uparrow}, \quad c^\dagger_{k\downarrow}\,) \begin{pmatrix} (\vec{\sigma}_{\uparrow\uparrow}\cdot\vec{H}) & (\vec{\sigma}_{\uparrow\downarrow}\cdot\vec{H}) \\ (\vec{\sigma}_{\downarrow\uparrow}\cdot\vec{H}) & (\vec{\sigma}_{\downarrow\downarrow}\cdot\vec{H}) \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \end{pmatrix} \\
&\quad + \frac{\mu_{\rm B}}{2}\sum_k (\, c_{-k\uparrow}, \quad c_{-k\downarrow}\,) \begin{pmatrix} (\vec{\sigma}_{\uparrow\uparrow}\cdot\vec{H}) & (\vec{\sigma}_{\downarrow\uparrow}\cdot\vec{H}) \\ (\vec{\sigma}_{\uparrow\downarrow}\cdot\vec{H}) & (\vec{\sigma}_{\downarrow\downarrow}\cdot\vec{H}) \end{pmatrix} \begin{pmatrix} c^\dagger_{-k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix} \\
&\quad - \frac{\mu_{\rm B}}{2}\sum_k \Big[(\vec{\sigma}_{\uparrow\uparrow}\cdot\vec{H}) + (\vec{\sigma}_{\downarrow\downarrow}\cdot\vec{H})\Big] \qquad (10) \\
&= -\frac{\mu_{\rm B}}{2}\sum_k (\, c^\dagger_{k\uparrow}, \quad c^\dagger_{k\downarrow}\,)\,(\hat{\vec{\sigma}}\cdot\vec{H}) \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \end{pmatrix} + \frac{\mu_{\rm B}}{2}\sum_k (\, c_{-k\uparrow}, \quad c_{-k\downarrow}\,)\,(\hat{\vec{\sigma}}^T\cdot\vec{H}) \begin{pmatrix} c^\dagger_{-k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix} \\
&\quad - \frac{\mu_{\rm B}}{2}\sum_k \Big[(\vec{\sigma}_{\uparrow\uparrow}\cdot\vec{H}) + (\vec{\sigma}_{\downarrow\downarrow}\cdot\vec{H})\Big]. \qquad (11)
\end{aligned}
$$

Then,

$$
H_{\rm Z} = \sum_k C^\dagger_k \frac{1}{2}\begin{pmatrix} -\mu_{\rm B}(\hat{\vec{\sigma}}\cdot\vec{H}) & 0 \\ 0 & \mu_{\rm B}(\hat{\vec{\sigma}}^T\cdot\vec{H}) \end{pmatrix} C_k, \qquad (12)
$$

where we have omitted the c-number term (i.e., the last term in Eq. (11)).

Hence,

$$
\begin{aligned}
H &= H_{\rm BCS} + H_{\rm Z} \qquad (13) \\
&= \sum_k C^\dagger_k \frac{1}{2}\begin{pmatrix} \xi_k\hat{\sigma}_0 & \hat{\Delta}_k \\ \hat{\Delta}^\dagger_k & -\xi_k\hat{\sigma}_0 \end{pmatrix} C_k + \sum_k C^\dagger_k \frac{1}{2}\begin{pmatrix} -\mu_{\rm B}(\hat{\vec{\sigma}}\cdot\vec{H}) & 0 \\ 0 & \mu_{\rm B}(\hat{\vec{\sigma}}^T\cdot\vec{H}) \end{pmatrix} C_k \qquad (14) \\
&= \sum_k C^\dagger_k \frac{1}{2}\begin{pmatrix} \xi_k\hat{\sigma}_0 - \mu_{\rm B}(\hat{\vec{\sigma}}\cdot\vec{H}) & \hat{\Delta}_k \\ \hat{\Delta}^\dagger_k & -\Big(\xi_k\hat{\sigma}_0 - \mu_{\rm B}(\hat{\vec{\sigma}}^T\cdot\vec{H})\Big) \end{pmatrix} C_k \qquad (15) \\
&\equiv \sum_k C^\dagger_k \frac{1}{2}\begin{pmatrix} \hat{T}_{1k} & \hat{\Delta}_k \\ \hat{\Delta}^\dagger_k & -\hat{T}_{2k} \end{pmatrix} C_k \qquad (16) \\
&\equiv \sum_k C^\dagger_k \check{\varepsilon}'_k C_k, \qquad (17)
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{T}_{1k} &= \xi_k\hat{\sigma}_0 - \mu_{\rm B}(\hat{\vec{\sigma}}\cdot\vec{H}), \qquad (18) \\
\hat{T}_{2k} &= \xi_k\hat{\sigma}_0 - \mu_{\rm B}(\hat{\vec{\sigma}}^T\cdot\vec{H}). \qquad (19)
\end{aligned}
$$

Here, $\hat{T}_{1,-k} = \hat{T}_{1k}$ and $\hat{T}_{2,-k} = \hat{T}_{2k}$ because $\xi_{-k} = \xi_k$. One can also confirm easily that $\hat{T}^*_{2k} = \hat{T}_{1k}$ because ξ_k and $\vec{H}$ are real.

Let us diagonalize it as in the problem **4.1**. By the unitary (Bogoliubov) transformation $\check{U}_k$,

$$
\begin{aligned}
H &= \sum_k \Big(C^\dagger_k\check{U}_k\Big)\Big(\check{U}^\dagger_k\check{\varepsilon}'_k\check{U}_k\Big)\Big(\check{U}^\dagger_k C_k\Big) \\
&= \sum_k A^\dagger_k \check{E}_k A_k, \qquad (20)
\end{aligned}
$$

where

$$
\begin{aligned}
\check{U}^\dagger_k\check{\varepsilon}'_k\check{U}_k &= \check{E}_k, \qquad (21) \\
A_k &= \check{U}^\dagger_k C_k. \qquad (22)
\end{aligned}
$$

From Eq. (21),

$$\breve{\varepsilon}'_k \breve{U}_k = \breve{U}_k \breve{E}_k. \tag{23}$$

The left hand side of this is

$$\breve{\varepsilon}'_k \breve{U}_k = \frac{1}{2}\begin{pmatrix} \hat{T}_{1k} & \hat{\Delta}_k \\ \hat{\Delta}_k^\dagger & -\hat{T}_{2k} \end{pmatrix}\begin{pmatrix} \hat{u}_k & \hat{v}_k \\ \hat{v}^*_{-k} & \hat{u}^*_{-k} \end{pmatrix} \tag{24}$$

$$= \frac{1}{2}\begin{pmatrix} \hat{T}_{1k}\hat{u}_k + \hat{\Delta}_k \hat{v}^*_{-k} & \hat{T}_{1k}\hat{v}_k + \hat{\Delta}_k \hat{u}^*_{-k} \\ \hat{\Delta}_k^\dagger \hat{u}_k - \hat{T}_{2k}\hat{v}^*_{-k} & \hat{\Delta}_k^\dagger \hat{v}_k - \hat{T}_{2k}\hat{u}^*_{-k} \end{pmatrix}. \tag{25}$$

The right hand side is

$$\breve{U}_k \breve{E}_k = \begin{pmatrix} \hat{u}_k & \hat{v}_k \\ \hat{v}^*_{-k} & \hat{u}^*_{-k} \end{pmatrix} \frac{1}{2}\begin{pmatrix} \hat{E}_k & 0 \\ 0 & -\hat{E}_{-k} \end{pmatrix} \tag{26}$$

$$= \frac{1}{2}\begin{pmatrix} \hat{u}_k \hat{E}_k & -\hat{v}_k \hat{E}_{-k} \\ \hat{v}^*_{-k}\hat{E}_k & -\hat{u}^*_{-k}\hat{E}_{-k} \end{pmatrix}. \tag{27}$$

Therefore, from $\breve{\varepsilon}'_k \breve{U}_k = \breve{U}_k \breve{E}_k$,

$$\frac{1}{2}\begin{pmatrix} \hat{T}_{1k}\hat{u}_k + \hat{\Delta}_k \hat{v}^*_{-k} & \hat{T}_{1k}\hat{v}_k + \hat{\Delta}_k \hat{u}^*_{-k} \\ \hat{\Delta}_k^\dagger \hat{u}_k - \hat{T}_{2k}\hat{v}^*_{-k} & \hat{\Delta}_k^\dagger \hat{v}_k - \hat{T}_{2k}\hat{u}^*_{-k} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \hat{u}_k \hat{E}_k & -\hat{v}_k \hat{E}_{-k} \\ \hat{v}^*_{-k}\hat{E}_k & -\hat{u}^*_{-k}\hat{E}_{-k} \end{pmatrix}. \tag{28}$$

We obtain the four equations,

$$\hat{T}_{1k}\hat{u}_k + \hat{\Delta}_k \hat{v}^*_{-k} = \hat{u}_k \hat{E}_k, \tag{29}$$

$$\hat{T}_{1k}\hat{v}_k + \hat{\Delta}_k \hat{u}^*_{-k} = -\hat{v}_k \hat{E}_{-k}, \tag{30}$$

$$\hat{\Delta}_k^\dagger \hat{u}_k - \hat{T}_{2k}\hat{v}^*_{-k} = \hat{v}^*_{-k}\hat{E}_k, \tag{31}$$

$$\hat{\Delta}_k^\dagger \hat{v}_k - \hat{T}_{2k}\hat{u}^*_{-k} = -\hat{u}^*_{-k}\hat{E}_{-k}. \tag{32}$$

From Eq. (31),

$$\hat{\Delta}_k^\dagger \hat{u}_k - \hat{T}_{2k}\hat{v}^*_{-k} = \hat{v}^*_{-k}\hat{E}_k, \tag{33}$$

$$\rightarrow \quad \hat{T}_{2k}\hat{v}^*_{-k} - \hat{\Delta}_k^\dagger \hat{u}_k = -\hat{v}^*_{-k}\hat{E}_k, \tag{34}$$

$$\rightarrow \quad \hat{T}_{2,-k}\hat{v}^*_k - \hat{\Delta}^\dagger_{-k}\hat{u}_{-k} = -\hat{v}^*_k \hat{E}_{-k}, \qquad (k \rightarrow -k) \tag{35}$$

$$\rightarrow \quad \hat{T}_{2k}\hat{v}^*_k + \left(\hat{\Delta}_k^T\right)^\dagger \hat{u}_{-k} = -\hat{v}^*_k \hat{E}_{-k}, \qquad (\hat{T}_{2,-k} = \hat{T}_{2k} \text{ and } \hat{\Delta}_{-k} = -\hat{\Delta}_k^T) \tag{36}$$

$$\rightarrow \quad \hat{T}_{2k}\hat{v}^*_k + \hat{\Delta}^*_k \hat{u}_{-k} = -\hat{v}^*_k \hat{E}_{-k}, \tag{37}$$

$$\rightarrow \quad \hat{T}^*_{2k}\hat{v}_k + \hat{\Delta}_k \hat{u}^*_{-k} = -\hat{v}_k \hat{E}_{-k} \tag{38}$$

$$\rightarrow \quad \hat{T}_{1k}\hat{v}_k + \hat{\Delta}_k \hat{u}^*_{-k} = -\hat{v}_k \hat{E}_{-k}. \qquad (\hat{T}^*_{2k} = \hat{T}_{1k}) \tag{39}$$

This is equivalent to Eq. (30), namely Eq. (31) is equivalent to Eq. (30).

From Eq. (32),

$$\hat{\Delta}_k^\dagger \hat{v}_k - \hat{T}_{2k}\hat{u}^*_{-k} = -\hat{u}^*_{-k}\hat{E}_{-k}, \tag{40}$$

$$\rightarrow \quad \hat{T}_{2k}\hat{u}^*_{-k} - \hat{\Delta}_k^\dagger \hat{v}_k = \hat{u}^*_{-k}\hat{E}_{-k}, \tag{41}$$

$$\rightarrow \quad \hat{T}_{2,-k}\hat{u}^*_k - \hat{\Delta}^\dagger_{-k}\hat{v}_{-k} = \hat{u}^*_k \hat{E}_k, \qquad (k \rightarrow -k) \tag{42}$$

$$\begin{aligned}
\rightarrow \qquad \hat{T}_{2k}\hat{u}_k^* + \left(\hat{\Delta}_k^T\right)^\dagger \hat{v}_{-k} &= \hat{u}_k^*\hat{E}_k, \qquad (\hat{T}_{2,-k} = \hat{T}_{2k} \text{ and } \hat{\Delta}_{-k} = -\hat{\Delta}_k^T) &(43)\\
\rightarrow \qquad \hat{T}_{2k}\hat{u}_k^* + \hat{\Delta}_k^*\hat{v}_{-k} &= \hat{u}_k^*\hat{E}_k, &(44)\\
\rightarrow \qquad \hat{T}_{2k}^*\hat{u}_k + \hat{\Delta}_k\hat{v}_{-k}^* &= \hat{u}_k\hat{E}_k &(45)\\
\rightarrow \qquad \hat{T}_{1k}\hat{u}_k + \hat{\Delta}_k\hat{v}_{-k}^* &= \hat{u}_k\hat{E}_k. \qquad (\hat{T}_{2k}^* = \hat{T}_{1k}) &(46)
\end{aligned}$$

This is equivalent to Eq. (29), namely Eq. (32) is equivalent to Eq. (29).

Thus, we obtain the two independent equations [Eq. (29) and (31)],

$$\begin{aligned}
\hat{T}_{1k}\hat{u}_k + \hat{\Delta}_k\hat{v}_{-k}^* &= \hat{u}_k\hat{E}_k, &(47)\\
\hat{\Delta}_k^\dagger\hat{u}_k - \hat{T}_{2k}\hat{v}_{-k}^* &= \hat{v}_{-k}^*\hat{E}_k. &(48)
\end{aligned}$$

Multiplying Eq. (47) by $\hat{\Delta}_k^\dagger$ from the left gives

$$\hat{\Delta}_k^\dagger\hat{T}_{1k}\hat{u}_k + \hat{\Delta}_k^\dagger\hat{\Delta}_k\hat{v}_{-k}^* = \hat{\Delta}_k^\dagger\hat{u}_k\hat{E}_k. \qquad (49)$$

Assuming the singlet state or the unitary triplet states $\hat{\Delta}_k\hat{\Delta}_k^\dagger = \hat{\Delta}_k^\dagger\hat{\Delta}_k = |\Delta_k|^2\hat{\sigma}_0$,

$$\begin{aligned}
\hat{\Delta}_k^\dagger\hat{T}_{1k}\hat{u}_k + |\Delta_k|^2\hat{v}_{-k}^* &= \hat{\Delta}_k^\dagger\hat{u}_k\hat{E}_k. &(50)\\
\rightarrow \qquad \hat{v}_{-k}^* &= \frac{1}{|\Delta_k|^2}\hat{\Delta}_k^\dagger\left(\hat{u}_k\hat{E}_k - \hat{T}_{1k}\hat{u}_k\right), &(51)
\end{aligned}$$

where $|\Delta_k|^2 \equiv \frac{1}{2}\mathrm{Tr}[\hat{\Delta}_k\hat{\Delta}_k^\dagger]$. Substituting this $\hat{v}_{-k}^*$ into Eq. (48),

$$\hat{\Delta}_k^\dagger\hat{u}_k - \hat{T}_{2k}\frac{1}{|\Delta_k|^2}\hat{\Delta}_k^\dagger\left(\hat{u}_k\hat{E}_k - \hat{T}_{1k}\hat{u}_k\right) = \frac{1}{|\Delta_k|^2}\hat{\Delta}_k^\dagger\left(\hat{u}_k\hat{E}_k - \hat{T}_{1k}\hat{u}_k\right)\hat{E}_k, \qquad (52)$$

$$\rightarrow \qquad |\Delta_k|^2\hat{\Delta}_k^\dagger\hat{u}_k - \hat{T}_{2k}\hat{\Delta}_k^\dagger\left(\hat{u}_k\hat{E}_k - \hat{T}_{1k}\hat{u}_k\right) = \hat{\Delta}_k^\dagger\left(\hat{u}_k\hat{E}_k - \hat{T}_{1k}\hat{u}_k\right)\hat{E}_k, \qquad (53)$$

$$\rightarrow \qquad |\Delta_k|^4\hat{u}_k - \hat{\Delta}_k\hat{T}_{2k}\hat{\Delta}_k^\dagger\left(\hat{u}_k\hat{E}_k - \hat{T}_{1k}\hat{u}_k\right) = |\Delta_k|^2\left(\hat{u}_k\hat{E}_k - \hat{T}_{1k}\hat{u}_k\right)\hat{E}_k, \qquad (54)$$

From now on, let us assume $\vec{H} = (0, 0, H_z) \parallel \hat{z}$. Then,

$$\hat{T}_{1k} = \hat{T}_{2k} = \xi_k\hat{\sigma}_0 - \mu_{\mathrm{B}}H_z\hat{\sigma}_z. \qquad (55)$$

We also assume

$$\begin{aligned}
\hat{u}_k &= u_k^0\hat{\sigma}_0 + u_k^z\hat{\sigma}_z &(56)\\
&= \begin{pmatrix} u_k^0 + u_k^z & 0 \\ 0 & u_k^0 - u_k^z \end{pmatrix}. &(57)
\end{aligned}$$

Then,

$$\begin{aligned}
\hat{u}_k\hat{E}_k - \hat{T}_{1k}\hat{u}_k &= \begin{pmatrix} u_k^0 + u_k^z & 0 \\ 0 & u_k^0 - u_k^z \end{pmatrix}\begin{pmatrix} E_{k,+} & 0 \\ 0 & E_{k,-} \end{pmatrix} - (\xi_k\hat{\sigma}_0 - \mu_{\mathrm{B}}H_z\hat{\sigma}_z)(u_k^0\hat{\sigma}_0 + u_k^z\hat{\sigma}_z) \\
&\qquad (58)\\
&= \begin{pmatrix} (u_k^0 + u_k^z)E_{k,+} & 0 \\ 0 & (u_k^0 - u_k^z)E_{k,-} \end{pmatrix} - \left((\xi_k u_k^0 - \mu_{\mathrm{B}}H_z u_k^z)\hat{\sigma}_0 + (\xi_k u_k^z - \mu_{\mathrm{B}}H_z u_k^0)\hat{\sigma}_z\right)
\end{aligned}$$

$$(59)$$

$$
\begin{aligned}
&= \begin{pmatrix} (u_k^0+u_k^z)E_{k,+} & 0 \\ 0 & (u_k^0-u_k^z)E_{k,-} \end{pmatrix} \\
&\quad - \begin{pmatrix} (\xi_k u_k^0 - \mu_{\rm B}H_z u_k^z) + (\xi_k u_k^z - \mu_{\rm B}H_z u_k^0) & 0 \\ 0 & (\xi_k u_k^0 - \mu_{\rm B}H_z u_k^z) - (\xi_k u_k^z - \mu_{\rm B}H_z u_k^0) \end{pmatrix} && (60) \\
&= \begin{pmatrix} (u_k^0+u_k^z)E_{k,+} & 0 \\ 0 & (u_k^0-u_k^z)E_{k,-} \end{pmatrix} \\
&\quad - \begin{pmatrix} (\xi_k - \mu_{\rm B}H_z)u_k^0 + (\xi_k - \mu_{\rm B}H_z)u_k^z & 0 \\ 0 & (\xi_k + \mu_{\rm B}H_z)u_k^0 + (-\xi_k - \mu_{\rm B}H_z)u_k^z \end{pmatrix} && (61) \\
&= \begin{pmatrix} (u_k^0+u_k^z)E_{k,+} & 0 \\ 0 & (u_k^0-u_k^z)E_{k,-} \end{pmatrix} \\
&\quad - \begin{pmatrix} (u_k^0+u_k^z)(\xi_k - \mu_{\rm B}H_z) & 0 \\ 0 & (u_k^0-u_k^z)(\xi_k + \mu_{\rm B}H_z) \end{pmatrix} && (62) \\
&= \begin{pmatrix} (u_k^0+u_k^z)\Big(E_{k,+} - (\xi_k - \mu_{\rm B}H_z)\Big) & 0 \\ 0 & (u_k^0-u_k^z)\Big(E_{k,-} - (\xi_k + \mu_{\rm B}H_z)\Big) \end{pmatrix} && (63) \\
&\equiv \begin{pmatrix} (u_k^0+u_k^z)\varepsilon_{k,+} & 0 \\ 0 & (u_k^0-u_k^z)\varepsilon_{k,-} \end{pmatrix}. && (64)
\end{aligned}
$$

And,

$$(\hat{u}_k\hat{E}_k - \hat{T}_{1k}\hat{u}_k)\hat{E}_k = \begin{pmatrix} (u_k^0+u_k^z)\varepsilon_{k,+}E_{k,+} & 0 \\ 0 & (u_k^0-u_k^z)\varepsilon_{k,-}E_{k,-} \end{pmatrix}. \quad (65)$$

Let us consider $\hat{\Delta}_k\hat{T}_{2k}$.

$$
\begin{aligned}
\hat{\Delta}_k\hat{T}_{2k} &= \hat{\Delta}_k\hat{T}_{1k} && (66) \\
&= \Big(\Psi\hat{\sigma}_0 + d_x\hat{\sigma}_x + d_y\hat{\sigma}_y + d_z\hat{\sigma}_z\Big)i\hat{\sigma}_y\Big(\xi_k\hat{\sigma}_0 - \mu_{\rm B}H_z\hat{\sigma}_z\Big) && (67) \\
&= \Big(\Psi i\hat{\sigma}_y - d_x\hat{\sigma}_z + id_y\hat{\sigma}_0 + d_z\hat{\sigma}_x\Big)\Big(\xi_k\hat{\sigma}_0 - \mu_{\rm B}H_z\hat{\sigma}_z\Big) && (68) \\
&= \xi_k\Big(\Psi i\hat{\sigma}_y - d_x\hat{\sigma}_z + id_y\hat{\sigma}_0 + d_z\hat{\sigma}_x\Big) - \mu_{\rm B}H_z\Big(-\Psi\hat{\sigma}_x - d_x\hat{\sigma}_0 + id_y\hat{\sigma}_z - id_z\hat{\sigma}_y\Big) && (69) \\
&= \xi_k\Big(\Psi i\hat{\sigma}_y - d_x\hat{\sigma}_z + id_y\hat{\sigma}_0 + d_z\hat{\sigma}_x\Big) - \mu_{\rm B}H_z\Big(-\Psi(i\hat{\sigma}_z\hat{\sigma}_y) - d_x(\hat{\sigma}_z\hat{\sigma}_z) + id_y\hat{\sigma}_z - id_z(-i\hat{\sigma}_z\hat{\sigma}_x)\Big) && (70) \\
&= \xi_k\Big(\Psi i\hat{\sigma}_y - d_x\hat{\sigma}_z + id_y\hat{\sigma}_0 + d_z\hat{\sigma}_x\Big) - \mu_{\rm B}H_z\hat{\sigma}_z\Big(-\Psi(i\hat{\sigma}_y) - d_x\hat{\sigma}_z + id_y\hat{\sigma}_0 - id_z(-i\hat{\sigma}_x)\Big) && (71) \\
&= \xi_k\Big(\Psi i\hat{\sigma}_y - d_x\hat{\sigma}_z + id_y\hat{\sigma}_0 + d_z\hat{\sigma}_x\Big) + \mu_{\rm B}H_z\hat{\sigma}_z\Big(\Psi i\hat{\sigma}_y + d_x\hat{\sigma}_z - id_y\hat{\sigma}_0 + d_z\hat{\sigma}_x\Big) && (72) \\
&= \xi_k\hat{\Delta}_k + \mu_{\rm B}H_z\hat{\sigma}_z\Big(\Psi - d_x\hat{\sigma}_x - d_y\hat{\sigma}_y + d_z\hat{\sigma}_z\Big)i\hat{\sigma}_y. && (73)
\end{aligned}
$$

In the case of the singlet state,

$$
\begin{aligned}
\hat{\Delta}_k\hat{T}_{2k} &= \xi_k(\Psi i\hat{\sigma}_y) + \mu_{\rm B}H_z\hat{\sigma}_z(\Psi i\hat{\sigma}_y) && (74) \\
&= (\xi_k\hat{\sigma}_0 + \mu_{\rm B}H_z\hat{\sigma}_z)(\Psi i\hat{\sigma}_y) && (75) \\
&= (\xi_k\hat{\sigma}_0 + \mu_{\rm B}H_z\hat{\sigma}_z)\hat{\Delta}_k. && (76)
\end{aligned}
$$

$$
\begin{aligned}
\hat{\Delta}_k\hat{T}_{2k}\hat{\Delta}_k^\dagger &= (\xi_k\hat{\sigma}_0 + \mu_{\rm B}H_z\hat{\sigma}_z)\hat{\Delta}_k\hat{\Delta}_k^\dagger && (77) \\
&= (\xi_k\hat{\sigma}_0 + \mu_{\rm B}H_z\hat{\sigma}_z)|\Delta_k|^2. && (78) \\
&= |\Delta_k|^2\begin{pmatrix} \xi_k + \mu_{\rm B}H_z & 0 \\ 0 & \xi_k - \mu_{\rm B}H_z \end{pmatrix} && (79)
\end{aligned}
$$

$$\hat{\Delta}_k \hat{T}_{2k} \hat{\Delta}_k^\dagger \Big(\hat{u}_k \hat{E}_k - \hat{T}_{1k} \hat{u}_k \Big) = |\Delta_k|^2 \begin{pmatrix} \xi_k + \mu_{\rm B} H_z & 0 \\ 0 & \xi_k - \mu_{\rm B} H_z \end{pmatrix} \begin{pmatrix} (u_k^0 + u_k^z)\varepsilon_{k,+} & 0 \\ 0 & (u_k^0 - u_k^z)\varepsilon_{k,-} \end{pmatrix} \tag{80}$$

$$= |\Delta_k|^2 \begin{pmatrix} (u_k^0 + u_k^z)(\xi_k + \mu_{\rm B} H_z)\varepsilon_{k,+} & 0 \\ 0 & (u_k^0 - u_k^z)(\xi_k - \mu_{\rm B} H_z)\varepsilon_{k,-} \end{pmatrix} \tag{81}$$

Therefore, considering Eq. (54):

$$|\Delta_k|^4 \hat{u}_k - \hat{\Delta}_k \hat{T}_{2k} \hat{\Delta}_k^\dagger \Big(\hat{u}_k \hat{E}_k - \hat{T}_{1k} \hat{u}_k \Big) = |\Delta_k|^2 \Big(\hat{u}_k \hat{E}_k - \hat{T}_{1k} \hat{u}_k \Big) \hat{E}_k, \tag{82}$$

we obtain two equations

$$|\Delta_k|^2 - (\xi_k + \mu_{\rm B} H_z)\varepsilon_{k,+} = \varepsilon_{k,+} E_{k,+}, \tag{83}$$

$$|\Delta_k|^2 - (\xi_k - \mu_{\rm B} H_z)\varepsilon_{k,-} = \varepsilon_{k,-} E_{k,-}. \tag{84}$$

$$\rightarrow \quad |\Delta_k|^2 - (\xi_k + \mu_{\rm B} H_z)\Big(E_{k,+} - (\xi_k - \mu_{\rm B} H_z)\Big) = \Big(E_{k,+} - (\xi_k - \mu_{\rm B} H_z)\Big) E_{k,+}, \tag{85}$$

$$|\Delta_k|^2 - (\xi_k - \mu_{\rm B} H_z)\Big(E_{k,-} - (\xi_k + \mu_{\rm B} H_z) = \Big(E_{k,-} - (\xi_k + \mu_{\rm B} H_z) E_{k,-}. \tag{86}$$

$$\rightarrow \quad |\Delta_k|^2 - (\xi_k + \mu_{\rm B} H_z) E_{k,+} + \xi_k^2 - (\mu_{\rm B} H_z)^2 = E_{k,+}^2 - (\xi_k - \mu_{\rm B} H_z) E_{k,+}, \tag{87}$$

$$|\Delta_k|^2 - (\xi_k - \mu_{\rm B} H_z) E_{k,-} + \xi_k^2 - (\mu_{\rm B} H_z)^2 = E_{k,-}^2 - (\xi_k + \mu_{\rm B} H_z) E_{k,-}. \tag{88}$$

$$\rightarrow \quad E_{k,+}^2 + 2\mu_{\rm B} H_z E_{k,+} - \Big(\xi_k^2 + |\Delta_k|^2 - (\mu_{\rm B} H_z)^2\Big) = 0, \tag{89}$$

$$E_{k,-}^2 - 2\mu_{\rm B} H_z E_{k,-} - \Big(\xi_k^2 + |\Delta_k|^2 - (\mu_{\rm B} H_z)^2\Big) = 0. \tag{90}$$

The solutions are, respectively,

$$E_{k,+} = -\mu_{\rm B} H_z \pm \sqrt{\xi_k^2 + |\Delta_k|^2} = -\mu_{\rm B} H_z \pm E_{0k}, \tag{91}$$

$$E_{k,-} = \mu_{\rm B} H_z \pm \sqrt{\xi_k^2 + |\Delta_k|^2} = \mu_{\rm B} H_z \pm E_{0k}, \tag{92}$$

where we have defined

$$E_{0k} \equiv \sqrt{\xi_k^2 + |\Delta_k|^2}. \tag{93}$$

One can assign the above four values to the four eigen values (i.e., the matrix elements of $\check{E}_k$) as follows. ($|\Delta_{-k}| = |\Delta_k|$, and thus $E_{0,-k} = E_{0k}$.)

$$\check{E}_k = \begin{pmatrix} E_{k,+} & 0 & 0 & 0 \\ 0 & E_{k,-} & 0 & 0 \\ 0 & 0 & -E_{-k,+} & 0 \\ 0 & 0 & 0 & -E_{-k,-} \end{pmatrix} \tag{94}$$

$$= \begin{pmatrix} E_{0k} - \mu_{\rm B} H_z & 0 & 0 & 0 \\ 0 & E_{0k} + \mu_{\rm B} H_z & 0 & 0 \\ 0 & 0 & -E_{0k} + \mu_{\rm B} H_z & 0 \\ 0 & 0 & 0 & -E_{0k} - \mu_{\rm B} H_z \end{pmatrix}. \tag{95}$$

In the case of the unitary triplet state with $\vec{d_k} \perp \vec{H}$, (i.e., $\vec{d} = (d_x, d_y, 0)$), from Eq. (73),

$$\begin{aligned}\hat{\Delta}_k \hat{T}_{2k} &= \xi_k \hat{\Delta}_k - \mu_{\rm B} H_z \hat{\sigma}_z \Big(d_x \hat{\sigma}_x + d_y \hat{\sigma}_y \Big) i \hat{\sigma}_y. &(96)\\ &= \xi_k \hat{\Delta}_k - \mu_{\rm B} H_z \hat{\sigma}_z \hat{\Delta}_k &(97)\\ &= (\xi_k \hat{\sigma}_0 - \mu_{\rm B} H_z \hat{\sigma}_z) \hat{\Delta}_k. &(98)\end{aligned}$$

$$\begin{aligned}\hat{\Delta}_k \hat{T}_{2k} \hat{\Delta}_k^\dagger &= (\xi_k \hat{\sigma}_0 - \mu_{\rm B} H_z \hat{\sigma}_z) \hat{\Delta}_k \hat{\Delta}_k^\dagger &(99)\\ &= (\xi_k \hat{\sigma}_0 - \mu_{\rm B} H_z \hat{\sigma}_z) |\Delta_k|^2. &(100)\\ &= |\Delta_k|^2 \begin{pmatrix} \xi_k - \mu_{\rm B} H_z & 0 \\ 0 & \xi_k + \mu_{\rm B} H_z \end{pmatrix} &(101)\end{aligned}$$

$$\begin{aligned}\hat{\Delta}_k \hat{T}_{2k} \hat{\Delta}_k^\dagger \Big(\hat{u}_k \hat{E}_k - \hat{T}_{1k} \hat{u}_k \Big) &= |\Delta_k|^2 \begin{pmatrix} \xi_k - \mu_{\rm B} H_z & 0 \\ 0 & \xi_k + \mu_{\rm B} H_z \end{pmatrix} \begin{pmatrix} (u_k^0 + u_k^z)\varepsilon_{k,+} & 0 \\ 0 & (u_k^0 - u_k^z)\varepsilon_{k,-} \end{pmatrix} &(102)\\ &= |\Delta_k|^2 \begin{pmatrix} (u_k^0 + u_k^z)(\xi_k - \mu_{\rm B} H_z)\varepsilon_{k,+} & 0 \\ 0 & (u_k^0 - u_k^z)(\xi_k + \mu_{\rm B} H_z)\varepsilon_{k,-} \end{pmatrix}. &(103)\end{aligned}$$

Therefore, considering Eq. (54):

$$|\Delta_k|^4 \hat{u}_k - \hat{\Delta}_k \hat{T}_{2k} \hat{\Delta}_k^\dagger \Big(\hat{u}_k \hat{E}_k - \hat{T}_{1k} \hat{u}_k \Big) = |\Delta_k|^2 \Big(\hat{u}_k \hat{E}_k - \hat{T}_{1k} \hat{u}_k \Big) \hat{E}_k, \quad (104)$$

we obtain two equations

$$\begin{aligned}|\Delta_k|^2 - (\xi_k - \mu_{\rm B} H_z)\varepsilon_{k,+} &= \varepsilon_{k,+} E_{k,+}, &(105)\\ |\Delta_k|^2 - (\xi_k + \mu_{\rm B} H_z)\varepsilon_{k,-} &= \varepsilon_{k,-} E_{k,-}. &(106)\end{aligned}$$

$$\begin{aligned}\rightarrow \quad |\Delta_k|^2 - (\xi_k - \mu_{\rm B} H_z)\Big(E_{k,+} - (\xi_k - \mu_{\rm B} H_z) \Big) &= \Big(E_{k,+} - (\xi_k - \mu_{\rm B} H_z) \Big) E_{k,+}, &(107)\\ |\Delta_k|^2 - (\xi_k + \mu_{\rm B} H_z)\Big(E_{k,-} - (\xi_k + \mu_{\rm B} H_z) &= \Big(E_{k,-} - (\xi_k + \mu_{\rm B} H_z) E_{k,-}. &(108)\end{aligned}$$

$$\begin{aligned}\rightarrow \quad |\Delta_k|^2 - (\xi_k - \mu_{\rm B} H_z) E_{k,+} + (\xi_k - \mu_{\rm B} H_z)^2 &= E_{k,+}^2 - (\xi_k - \mu_{\rm B} H_z) E_{k,+}, &(109)\\ |\Delta_k|^2 - (\xi_k + \mu_{\rm B} H_z) E_{k,-} + (\xi_k + \mu_{\rm B} H_z)^2 &= E_{k,-}^2 - (\xi_k + \mu_{\rm B} H_z) E_{k,-}. &(110)\end{aligned}$$

$$\begin{aligned}\rightarrow \quad E_{k,+}^2 &= (\xi_k - \mu_{\rm B} H_z)^2 + |\Delta_k|^2, &(111)\\ E_{k,-}^2 &= (\xi_k + \mu_{\rm B} H_z)^2 + |\Delta_k|^2. &(112)\end{aligned}$$

We define

$$\begin{aligned}E_{k,+}^\perp &= \sqrt{(\xi_k - \mu_{\rm B} H_z)^2 + |\Delta_k|^2}, &(113)\\ E_{k,-}^\perp &= \sqrt{(\xi_k + \mu_{\rm B} H_z)^2 + |\Delta_k|^2}. &(114)\end{aligned}$$

Thus, we have obtained four eigen values (i.e., the matrix elements of $\check{E}_k$) as follows. ($|\Delta_{-k}| = |\Delta_k|$ and $\xi_{-k} = \xi_k$, and thus $E^{\perp}_{-k,\pm} = E^{\perp}_{k,\pm}$.)

$$\check{E}_k = \begin{pmatrix} E_{k,+} & 0 & 0 & 0 \\ 0 & E_{k,-} & 0 & 0 \\ 0 & 0 & -E_{-k,+} & 0 \\ 0 & 0 & 0 & -E_{-k,-} \end{pmatrix} \tag{115}$$

$$= \begin{pmatrix} E^{\perp}_{k,+} & 0 & 0 & 0 \\ 0 & E^{\perp}_{k,-} & 0 & 0 \\ 0 & 0 & -E^{\perp}_{k,+} & 0 \\ 0 & 0 & 0 & -E^{\perp}_{k,-} \end{pmatrix}. \tag{116}$$

In the case of the unitary triplet state with $\vec{d}_k \parallel \vec{H}$, (i.e., $\vec{d} = (0, 0, d_z)$), from Eq. (73),

$$\hat{\Delta}_k \hat{T}_{2k} = \xi_k \hat{\Delta}_k + \mu_{\rm B} H_z \hat{\sigma}_z \Big(d_z \hat{\sigma}_z \Big) i\hat{\sigma}_y \tag{117}$$

$$= (\xi_k \hat{\sigma}_0 + \mu_{\rm B} H_z \hat{\sigma}_z) \hat{\Delta}_k. \tag{118}$$

$$\hat{\Delta}_k \hat{T}_{2k} \hat{\Delta}^{\dagger}_k = (\xi_k \hat{\sigma}_0 + \mu_{\rm B} H_z \hat{\sigma}_z) \hat{\Delta}_k \hat{\Delta}^{\dagger}_k \tag{119}$$

$$= (\xi_k \hat{\sigma}_0 + \mu_{\rm B} H_z \hat{\sigma}_z) |\Delta_k|^2. \tag{120}$$

$$= |\Delta_k|^2 \begin{pmatrix} \xi_k + \mu_{\rm B} H_z & 0 \\ 0 & \xi_k - \mu_{\rm B} H_z \end{pmatrix} \tag{121}$$

$$\hat{\Delta}_k \hat{T}_{2k} \hat{\Delta}^{\dagger}_k \Big(\hat{u}_k \hat{E}_k - \hat{T}_{1k} \hat{u}_k \Big) = |\Delta_k|^2 \begin{pmatrix} \xi_k + \mu_{\rm B} H_z & 0 \\ 0 & \xi_k - \mu_{\rm B} H_z \end{pmatrix} \begin{pmatrix} (u^0_k + u^z_k)\varepsilon_{k,+} & 0 \\ 0 & (u^0_k - u^z_k)\varepsilon_{k,-} \end{pmatrix} \tag{122}$$

$$= |\Delta_k|^2 \begin{pmatrix} (u^0_k + u^z_k)(\xi_k + \mu_{\rm B} H_z)\varepsilon_{k,+} & 0 \\ 0 & (u^0_k - u^z_k)(\xi_k - \mu_{\rm B} H_z)\varepsilon_{k,-} \end{pmatrix}. \tag{123}$$

Therefore, considering Eq. (54):

$$|\Delta_k|^4 \hat{u}_k - \hat{\Delta}_k \hat{T}_{2k} \hat{\Delta}^{\dagger}_k \Big(\hat{u}_k \hat{E}_k - \hat{T}_{1k} \hat{u}_k \Big) = |\Delta_k|^2 \Big(\hat{u}_k \hat{E}_k - \hat{T}_{1k} \hat{u}_k \Big) \hat{E}_k, \tag{124}$$

we obtain two equations

$$|\Delta_k|^2 - (\xi_k + \mu_{\rm B} H_z)\varepsilon_{k,+} = \varepsilon_{k,+} E_{k,+}, \tag{125}$$

$$|\Delta_k|^2 - (\xi_k - \mu_{\rm B} H_z)\varepsilon_{k,-} = \varepsilon_{k,-} E_{k,-}. \tag{126}$$

$$\rightarrow \quad |\Delta_k|^2 - (\xi_k + \mu_{\rm B} H_z)\Big(E_{k,+} - (\xi_k - \mu_{\rm B} H_z)\Big) = \Big(E_{k,+} - (\xi_k - \mu_{\rm B} H_z)\Big) E_{k,+}, \tag{127}$$

$$|\Delta_k|^2 - (\xi_k - \mu_{\rm B} H_z)\Big(E_{k,-} - (\xi_k + \mu_{\rm B} H_z) = \Big(E_{k,-} - (\xi_k + \mu_{\rm B} H_z) E_{k,-}. \tag{128}$$

$$\rightarrow \quad |\Delta_k|^2 - (\xi_k + \mu_{\rm B} H_z) E_{k,+} + \xi_k^2 - (\mu_{\rm B} H_z)^2 = E^2_{k,+} - (\xi_k - \mu_{\rm B} H_z) E_{k,+}, \tag{129}$$

$$|\Delta_k|^2 - (\xi_k - \mu_{\rm B} H_z) E_{k,-} + \xi_k^2 - (\mu_{\rm B} H_z)^2 = E^2_{k,-} - (\xi_k + \mu_{\rm B} H_z) E_{k,-}. \tag{130}$$

$$\rightarrow \qquad E_{k,+}^2 + 2\mu_{\rm B}H_z E_{k,+} - \left(\xi_k^2 + |\Delta_k|^2 - (\mu_{\rm B}H_z)^2\right) = 0, \tag{131}$$

$$E_{k,-}^2 - 2\mu_{\rm B}H_z E_{k,-} - \left(\xi_k^2 + |\Delta_k|^2 - (\mu_{\rm B}H_z)^2\right) = 0. \tag{132}$$

The solutions are, respectively,

$$E_{k,+} = -\mu_{\rm B}H_z \pm \sqrt{\xi_k^2 + |\Delta_k|^2} = -\mu_{\rm B}H_z \pm E_{0k}, \tag{133}$$

$$E_{k,-} = \mu_{\rm B}H_z \pm \sqrt{\xi_k^2 + |\Delta_k|^2} = \mu_{\rm B}H_z \pm E_{0k}, \tag{134}$$

where we have defined

$$E_{0k} \equiv \sqrt{\xi_k^2 + |\Delta_k|^2}. \tag{135}$$

One can assign the above four values to the four eigen values (i.e., the matrix elements of $\check{E}_k$) as follows. ($|\Delta_{-k}| = |\Delta_k|$, and thus $E_{0,-k} = E_{0k}$.)

$$\check{E}_k = \begin{pmatrix} E_{k,+} & 0 & 0 & 0 \\ 0 & E_{k,-} & 0 & 0 \\ 0 & 0 & -E_{-k,+} & 0 \\ 0 & 0 & 0 & -E_{-k,-} \end{pmatrix} \tag{136}$$

$$= \begin{pmatrix} E_{0k} - \mu_{\rm B}H_z & 0 & 0 & 0 \\ 0 & E_{0k} + \mu_{\rm B}H_z & 0 & 0 \\ 0 & 0 & -E_{0k} + \mu_{\rm B}H_z & 0 \\ 0 & 0 & 0 & -E_{0k} - \mu_{\rm B}H_z \end{pmatrix}. \tag{137}$$

The result is quite the same as in the case of the singlet state.

In summary:

- In the singlet state, $\quad E_{k,\pm} = \sqrt{\xi_k^2 + |\Delta_k|^2} \mp \mu_{\rm B}H_z$.
- In the unitary triplet state with $\vec{d} \perp \vec{H}$, $\quad E_{k,\pm} = \sqrt{(\xi_k \mp \mu_{\rm B}H_z)^2 + |\Delta_k|^2}$.
- In the unitary triplet state with $\vec{d} \parallel \vec{H}$, $\quad E_{k,\pm} = \sqrt{\xi_k^2 + |\Delta_k|^2} \mp \mu_{\rm B}H_z$, which is the same result as in the singlet state.

8.2 Let us consider Eqs. (47) and (48):

$$\hat{T}_{1k}\hat{u}_k + \hat{\Delta}_k\hat{v}^*_{-k} = \hat{u}_k\hat{E}_k, \tag{138}$$
$$\hat{\Delta}^\dagger_k\hat{u}_k - \hat{T}_{2k}\hat{v}^*_{-k} = \hat{v}^*_{-k}\hat{E}_k. \tag{139}$$

We assume

$$\hat{u}_k = u^0_k\hat{\sigma}_0 + u^z_k\hat{\sigma}_z \tag{140}$$
$$= \begin{pmatrix} u^0_k + u^z_k & 0 \\ 0 & u^0_k - u^z_k \end{pmatrix}. \tag{141}$$

From Eq. (47) (i.e., Eq. (138)), we obtained Eq. (51):

$$\hat{v}^*_{-k} = \frac{1}{|\Delta_k|^2}\hat{\Delta}^\dagger_k\Big(\hat{u}_k\hat{E}_k - \hat{T}_{1k}\hat{u}_k\Big). \tag{142}$$

Using Eq. (64),

$$\hat{v}^*_{-k} = \frac{1}{|\Delta_k|^2}\hat{\Delta}^\dagger_k\begin{pmatrix} (u^0_k + u^z_k)\varepsilon_{k,+} & 0 \\ 0 & (u^0_k - u^z_k)\varepsilon_{k,-} \end{pmatrix} \tag{143}$$
$$= \frac{1}{|\Delta_k|^2}\hat{\Delta}^\dagger_k\begin{pmatrix} \varepsilon_{k,+} & 0 \\ 0 & \varepsilon_{k,-} \end{pmatrix}\hat{u}_k \tag{144}$$
$$\equiv \frac{1}{|\Delta_k|^2}\hat{\Delta}^\dagger_k\hat{\varepsilon}_{k,+-}\hat{u}_k. \tag{145}$$

$$\rightarrow \qquad \hat{v}_k = \frac{1}{|\Delta_{-k}|^2}\hat{\Delta}^T_{-k}\hat{\varepsilon}_{-k,+-}\hat{u}^*_{-k} \tag{146}$$
$$= \frac{1}{|\Delta_k|^2}(-\hat{\Delta}_k)\hat{\varepsilon}_{k,+-}\hat{u}^*_{-k} \tag{147}$$
$$= \frac{-1}{|\Delta_k|^2}\hat{\Delta}_k\hat{\varepsilon}_{k,+-}\hat{u}^*_{-k}. \tag{148}$$

$$\rightarrow \qquad \hat{v}^\dagger_k = \frac{-1}{|\Delta_k|^2}\hat{u}_{-k}\hat{\varepsilon}_{k,+-}\hat{\Delta}^\dagger_k. \tag{149}$$

On the other hand, because $\check{U}_k$ is a unitary matrix,

$$\check{1} = \check{U}_k\check{U}^\dagger_k \tag{150}$$
$$= \begin{pmatrix} \hat{u}_k & \hat{v}_k \\ \hat{v}^*_{-k} & \hat{u}^*_{-k} \end{pmatrix}\begin{pmatrix} \hat{u}^\dagger_k & \hat{v}^{*\dagger}_{-k} \\ \hat{v}^\dagger_k & \hat{u}^{*\dagger}_{-k} \end{pmatrix} \tag{151}$$
$$= \begin{pmatrix} \hat{u}_k & \hat{v}_k \\ \hat{v}^*_{-k} & \hat{u}^*_{-k} \end{pmatrix}\begin{pmatrix} \hat{u}^\dagger_k & \hat{v}^T_{-k} \\ \hat{v}^\dagger_k & \hat{u}^T_{-k} \end{pmatrix} \tag{152}$$
$$= \begin{pmatrix} \hat{u}_k\hat{u}^\dagger_k + \hat{v}_k\hat{v}^\dagger_k & \hat{u}_k\hat{v}^T_{-k} + \hat{v}_k\hat{u}^T_{-k} \\ \hat{v}^*_{-k}\hat{u}^\dagger_k + \hat{u}^*_{-k}\hat{v}^\dagger_k & \hat{v}^*_{-k}\hat{v}^T_{-k} + \hat{u}^*_{-k}\hat{u}^T_{-k} \end{pmatrix}. \tag{153}$$

Thus,

$$\hat{u}_k\hat{u}^\dagger_k + \hat{v}_k\hat{v}^\dagger_k = \hat{\sigma}_0, \tag{154}$$
$$\hat{v}^*_{-k}\hat{u}^\dagger_k + \hat{u}^*_{-k}\hat{v}^\dagger_k = 0. \tag{155}$$

Substituting Eqs. (145) and (149) into Eq. (155),

$$\frac{1}{|\Delta_k|^2}\hat{\Delta}_k^\dagger \hat{\varepsilon}_{k,+-}\hat{u}_k\hat{u}_k^\dagger + \hat{u}_{-k}^* \frac{-1}{|\Delta_k|^2}\hat{u}_{-k}\hat{\varepsilon}_{k,+-}\hat{\Delta}_k^\dagger = 0, \tag{156}$$

$$\rightarrow \qquad \frac{1}{|\Delta_k|^2}\hat{\Delta}_k^\dagger \hat{\varepsilon}_{k,+-}\hat{u}_k\hat{u}_k^\dagger \hat{\Delta}_k - \hat{u}_{-k}^* \hat{u}_{-k}\hat{\varepsilon}_{k,+-} = 0, \tag{157}$$

$$\rightarrow \qquad \hat{u}_{-k}^* \hat{u}_{-k}\hat{\varepsilon}_{k,+-} = \frac{1}{|\Delta_k|^2}\hat{\Delta}_k^\dagger \hat{\varepsilon}_{k,+-}\hat{u}_k\hat{u}_k^\dagger \hat{\Delta}_k. \tag{158}$$

Substituting Eqs. (148) and (149) into Eq. (154),

$$\hat{u}_k\hat{u}_k^\dagger + \left(\frac{-1}{|\Delta_k|^2}\hat{\Delta}_k \hat{\varepsilon}_{k,+-}\hat{u}_{-k}^*\right)\left(\frac{-1}{|\Delta_k|^2}\hat{u}_{-k}\hat{\varepsilon}_{k,+-}\hat{\Delta}_k^\dagger\right) \;=\; \hat{\sigma}_0, \tag{159}$$

$$\rightarrow \qquad \hat{u}_k\hat{u}_k^\dagger + \frac{1}{|\Delta_k|^4}\hat{\Delta}_k \hat{\varepsilon}_{k,+-}(\hat{u}_{-k}^*\hat{u}_{-k}\hat{\varepsilon}_{k,+-})\hat{\Delta}_k^\dagger \;=\; \hat{\sigma}_0, \tag{160}$$

$$\rightarrow \qquad \hat{u}_k\hat{u}_k^\dagger + \frac{1}{|\Delta_k|^4}\hat{\Delta}_k \hat{\varepsilon}_{k,+-}\left(\frac{1}{|\Delta_k|^2}\hat{\Delta}_k^\dagger \hat{\varepsilon}_{k,+-}\hat{u}_k\hat{u}_k^\dagger \hat{\Delta}_k\right)\hat{\Delta}_k^\dagger \;=\; \hat{\sigma}_0, \tag{161}$$

$$\rightarrow \qquad \hat{u}_k\hat{u}_k^\dagger + \frac{1}{|\Delta_k|^4}(\hat{\Delta}_k \hat{\varepsilon}_{k,+-}\hat{\Delta}_k^\dagger \hat{\varepsilon}_{k,+-})\hat{u}_k\hat{u}_k^\dagger \;=\; \hat{\sigma}_0. \tag{162}$$

Here,

$$\hat{u}_k\hat{u}_k^\dagger \;=\; (u_k^0\hat{\sigma}_0 + u_k^z\hat{\sigma}_z)(u_k^{0*}\hat{\sigma}_0 + u_k^{z*}\hat{\sigma}_z) \tag{163}$$

$$\;=\; (|u_k^0|^2 + |u_k^z|^2)\hat{\sigma}_0 + (u_k^0 u_k^{z*} + u_k^z u_k^{0*})\hat{\sigma}_z. \tag{164}$$

In the case of the singlet state,

$$\hat{\Delta}_k^\dagger \hat{\varepsilon}_{k,+-} = \begin{pmatrix} 0 & -\Psi_k^* \\ \Psi_k^* & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_{k,+} & 0 \\ 0 & \varepsilon_{k,-} \end{pmatrix} \tag{165}$$

$$= \begin{pmatrix} 0 & -\Psi_k^* \varepsilon_{k,-} \\ \Psi_k^* \varepsilon_{k,+} & 0 \end{pmatrix}. \tag{166}$$

$$\hat{\varepsilon}_{k,+-} \hat{\Delta}_k^\dagger \hat{\varepsilon}_{k,+-} = \begin{pmatrix} \varepsilon_{k,+} & 0 \\ 0 & \varepsilon_{k,-} \end{pmatrix} \begin{pmatrix} 0 & -\Psi_k^* \varepsilon_{k,-} \\ \Psi_k^* \varepsilon_{k,+} & 0 \end{pmatrix} \tag{167}$$

$$= \begin{pmatrix} 0 & -\Psi_k^* \varepsilon_{k,+} \varepsilon_{k,-} \\ \Psi_k^* \varepsilon_{k,+} \varepsilon_{k,-} & 0 \end{pmatrix}. \tag{168}$$

$$\hat{\Delta}_k \hat{\varepsilon}_{k,+-} \hat{\Delta}_k^\dagger \hat{\varepsilon}_{k,+-} = \begin{pmatrix} 0 & \Psi_k \\ -\Psi_k & 0 \end{pmatrix} \begin{pmatrix} 0 & -\Psi_k^* \varepsilon_{k,+} \varepsilon_{k,-} \\ \Psi_k^* \varepsilon_{k,+} \varepsilon_{k,-} & 0 \end{pmatrix} \tag{169}$$

$$= \begin{pmatrix} |\Psi_k|^2 \varepsilon_{k,+} \varepsilon_{k,-} & 0 \\ 0 & |\Psi_k|^2 \varepsilon_{k,+} \varepsilon_{k,-} \end{pmatrix} \tag{170}$$

$$= |\Psi_k|^2 \varepsilon_{k,+} \varepsilon_{k,-} \hat{\sigma}_0 \tag{171}$$

$$= |\Delta_k|^2 \varepsilon_{k,+} \varepsilon_{k,-} \hat{\sigma}_0. \tag{172}$$

Substituting this into Eq. (162),

$$\hat{u}_k \hat{u}_k^\dagger + \frac{1}{|\Delta_k|^2} \varepsilon_{k,+} \varepsilon_{k,-} \hat{u}_k \hat{u}_k^\dagger = \hat{\sigma}_0, \tag{173}$$

$$\rightarrow \qquad \hat{u}_k \hat{u}_k^\dagger = \left[1 + \frac{\varepsilon_{k,+} \varepsilon_{k,-}}{|\Delta_k|^2} \right]^{-1} \hat{\sigma}_0 \tag{174}$$

$$= \left(\frac{1}{|\Delta_k|^2} \right)^{-1} \left[|\Delta_k|^2 + \Big\{ E_{k,+} - (\xi_k - \mu_{\mathrm{B}} H_z) \Big\} \Big\{ E_{k,-} - (\xi_k + \mu_{\mathrm{B}} H_z) \Big\} \right]^{-1} \hat{\sigma}_0 \tag{175}$$

$$= |\Delta_k|^2 \left[|\Delta_k|^2 + \Big\{ E_{0k} - \xi_k \Big\} \Big\{ E_{0k} - \xi_k \Big\} \right]^{-1} \hat{\sigma}_0 \tag{176}$$

$$= |\Delta_k|^2 \Big[|\Delta_k|^2 + \xi_k^2 + E_{0k}^2 - 2E_{0k}\xi_k \Big]^{-1} \hat{\sigma}_0 \tag{177}$$

$$= |\Delta_k|^2 \Big[2E_{0k}^2 - 2E_{0k}\xi_k \Big]^{-1} \hat{\sigma}_0 \tag{178}$$

$$= \frac{|\Delta_k|^2}{2E_{0k}(E_{0k} - \xi_k)} \hat{\sigma}_0 \tag{179}$$

$$= \frac{|\Psi_k|^2}{2E_{0k}(E_{0k} - \xi_k)} \hat{\sigma}_0 \tag{180}$$

$$= \frac{|\Psi_k|^2 (E_{0k} + \xi_k)}{2E_{0k}(E_{0k} - \xi_k)(E_{0k} + \xi_k)} \hat{\sigma}_0 \tag{181}$$

$$= \frac{|\Psi_k|^2 (E_{0k} + \xi_k)}{2E_{0k}(E_{0k}^2 - \xi_k^2)} \hat{\sigma}_0 \tag{182}$$

$$= \frac{|\Psi_k|^2 (E_{0k} + \xi_k)}{2E_{0k} |\Psi_k|^2} \hat{\sigma}_0 \tag{183}$$

$$= \frac{E_{0k} + \xi_k}{2E_{0k}} \hat{\sigma}_0. \tag{184}$$

From this and Eq. (164),

$$
\begin{aligned}
(u_k^0 u_k^{z*} + u_k^z u_k^{0*}) &= 0, &(185)\\
|u_k^0|^2 + |u_k^z|^2 &= \frac{E_{0k}+\xi_k}{2E_{0k}}. &(186)
\end{aligned}
$$

Assuming $u_k^z = 0$ and u_k^0 is real, one can determine u_k^0 as

$$
\begin{aligned}
u_k^0 &= \sqrt{\frac{E_{0k}+\xi_k}{2E_{0k}}} &(187)\\
&= \frac{E_{0k}+\xi_k}{\sqrt{2E_{0k}(E_{0k}+\xi_k)}}. &(188)
\end{aligned}
$$

Then,

$$
\begin{aligned}
\hat{u}_k &= u_k^0 \hat{\sigma}_0 &(189)\\
&= \frac{E_{0k}+\xi_k}{\sqrt{2E_{0k}(E_{0k}+\xi_k)}}\hat{\sigma}_0. &(190)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
u_{-k}^{0*} &= \frac{E_{0,-k}+\xi_{-k}}{\sqrt{2E_{0,-k}(E_{0,-k}+\xi_{-k})}} &(191)\\
&= \frac{E_{0k}+\xi_k}{\sqrt{2E_{0k}(E_{0k}+\xi_k)}}. &(192)
\end{aligned}
$$

And

$$
\hat{u}_{-k}^* = \frac{E_{0k}+\xi_k}{\sqrt{2E_{0k}(E_{0k}+\xi_k)}}\hat{\sigma}_0. \qquad (193)
$$

Substituting this into Eq. (148),

$$
\begin{aligned}
\hat{v}_k &= \frac{-1}{|\Delta_k|^2}\hat{\Delta}_k \hat{\varepsilon}_{k,+-}\hat{u}_{-k}^* &(194)\\
&= \frac{-1}{|\Delta_k|^2}\hat{\Delta}_k \hat{\varepsilon}_{k,+-}\frac{E_{0k}+\xi_k}{\sqrt{2E_{0k}(E_{0k}+\xi_k)}}\hat{\sigma}_0 &(195)\\
&= \frac{-(E_{0k}+\xi_k)}{\sqrt{2E_{0k}(E_{0k}+\xi_k)}}\frac{1}{|\Psi_k|^2}(\Psi_k i\hat{\sigma}_y)\begin{pmatrix}\varepsilon_{k,+} & 0\\ 0 & \varepsilon_{k,-}\end{pmatrix} &(196)\\
&= \frac{-(E_{0k}+\xi_k)}{\sqrt{2E_{0k}(E_{0k}+\xi_k)}}\frac{\Psi_k}{|\Psi_k|^2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\varepsilon_{k,+} & 0\\ 0 & \varepsilon_{k,-}\end{pmatrix} &(197)\\
&= \frac{-(E_{0k}+\xi_k)}{\sqrt{2E_{0k}(E_{0k}+\xi_k)}}\frac{\Psi_k}{|\Psi_k|^2}\begin{pmatrix}0 & \varepsilon_{k,-}\\ -\varepsilon_{k,+} & 0\end{pmatrix} &(198)\\
&= \frac{-(E_{0k}+\xi_k)}{\sqrt{2E_{0k}(E_{0k}+\xi_k)}}\frac{\Psi_k}{|\Psi_k|^2}\begin{pmatrix}0 & E_{k,+}-(\xi_k-\mu_{\rm B}H_z)\\ -(E_{k,-}-(\xi_k+\mu_{\rm B}H_z)) & 0\end{pmatrix} &(199)\\
&= \frac{-(E_{0k}+\xi_k)}{\sqrt{2E_{0k}(E_{0k}+\xi_k)}}\frac{\Psi_k}{|\Psi_k|^2}\begin{pmatrix}0 & (E_{0k}-\xi_k)\\ -(E_{0,k}-\xi_k) & 0\end{pmatrix} &(200)
\end{aligned}
$$

$$= \frac{-(E_{0k}+\xi_k)(E_{0k}-\xi_k)}{\sqrt{2E_{0k}(E_{0k}+\xi_k)}}\frac{\Psi_k}{|\Psi_k|^2}i\hat{\sigma}_y \tag{201}$$

$$= \frac{-(E_{0,k}^2-\xi_k^2)}{\sqrt{2E_{0k}(E_{0k}+\xi_k)}}\frac{\Psi_k}{|\Psi_k|^2}i\hat{\sigma}_y \tag{202}$$

$$= \frac{-|\Psi_k|^2}{\sqrt{2E_{0k}(E_{0k}+\xi_k)}}\frac{\Psi_k}{|\Psi_k|^2}i\hat{\sigma}_y \tag{203}$$

$$= \frac{-\Psi_k}{\sqrt{2E_{0k}(E_{0k}+\xi_k)}}i\hat{\sigma}_y. \tag{204}$$

Hence,

$$u_{k\uparrow\uparrow} = u_{k\downarrow\downarrow} = \frac{E_{0k}+\xi_k}{\sqrt{2E_{0k}(E_{0k}+\xi_k)}}, \tag{205}$$

$$u_{k\uparrow\downarrow} = u_{k\downarrow\uparrow} = 0, \tag{206}$$

$$v_{k\uparrow\uparrow} = v_{k\downarrow\downarrow} = 0, \tag{207}$$

$$v_{k\uparrow\downarrow} = -v_{k\downarrow\uparrow} = \frac{-\Psi_k}{\sqrt{2E_{0k}(E_{0k}+\xi_k)}}. \tag{208}$$

Here,

$$E_{0k} = \sqrt{\xi_k^2+|\Psi_k|^2}. \tag{209}$$

These results for u and v are the same as those in the zero magnetic-field case (Serie 4).

In the case of the unitary triplet state with $\vec{d} \perp \vec{H}$ (i.e., $\vec{d} = (d_x, d_y, 0)$),

$$\hat{\Delta}_k^\dagger \hat{\varepsilon}_{k,+-} = \begin{pmatrix} -d_x^* - id_y^* & 0 \\ 0 & d_x^* - id_y^* \end{pmatrix} \begin{pmatrix} \varepsilon_{k,+} & 0 \\ 0 & \varepsilon_{k,-} \end{pmatrix} \tag{210}$$

$$= \begin{pmatrix} -(d_x^* + id_y^*)\varepsilon_{k,+} & 0 \\ 0 & (d_x^* - id_y^*)\varepsilon_{k,-} \end{pmatrix}. \tag{211}$$

$$\hat{\varepsilon}_{k,+-} \hat{\Delta}_k^\dagger \hat{\varepsilon}_{k,+-} = \begin{pmatrix} \varepsilon_{k,+} & 0 \\ 0 & \varepsilon_{k,-} \end{pmatrix} \begin{pmatrix} -(d_x^* + id_y^*)\varepsilon_{k,+} & 0 \\ 0 & (d_x^* - id_y^*)\varepsilon_{k,-} \end{pmatrix} \tag{212}$$

$$= \begin{pmatrix} -(d_x^* + id_y^*)\varepsilon_{k,+}^2 & 0 \\ 0 & (d_x^* - id_y^*)\varepsilon_{k,-}^2 \end{pmatrix}. \tag{213}$$

$$\hat{\Delta}_k \hat{\varepsilon}_{k,+-} \hat{\Delta}_k^\dagger \hat{\varepsilon}_{k,+-} = \begin{pmatrix} -d_x + id_y & 0 \\ 0 & d_x + id_y \end{pmatrix} \begin{pmatrix} -(d_x^* + id_y^*)\varepsilon_{k,+}^2 & 0 \\ 0 & (d_x^* - id_y^*)\varepsilon_{k,-}^2 \end{pmatrix} \tag{214}$$

$$= \begin{pmatrix} (|d_x|^2 + |d_y|^2)\varepsilon_{k,+}^2 & 0 \\ 0 & (|d_x|^2 + |d_y|^2)\varepsilon_{k,-}^2 \end{pmatrix}, \qquad (\vec{d} \times \vec{d}^* = 0) \tag{215}$$

$$= |\Delta_k|^2 \begin{pmatrix} \varepsilon_{k,+}^2 & 0 \\ 0 & \varepsilon_{k,-}^2 \end{pmatrix}. \tag{216}$$

Substituting this into Eq. (162),

$$\hat{u}_k \hat{u}_k^\dagger + \frac{1}{|\Delta_k|^2} \begin{pmatrix} \varepsilon_{k,+}^2 & 0 \\ 0 & \varepsilon_{k,-}^2 \end{pmatrix} \hat{u}_k \hat{u}_k^\dagger = \hat{\sigma}_0. \tag{217}$$

$$\rightarrow \quad \begin{pmatrix} 1 + \varepsilon_{k,+}^2/|\Delta_k|^2 & 0 \\ 0 & 1 + \varepsilon_{k,-}^2/|\Delta_k|^2 \end{pmatrix} \hat{u}_k \hat{u}_k^\dagger = \hat{\sigma}_0, \tag{218}$$

$$\rightarrow \quad \hat{u}_k \hat{u}_k^\dagger = \frac{1}{(1 + \varepsilon_{k,+}^2/|\Delta_k|^2)(1 + \varepsilon_{k,-}^2/|\Delta_k|^2)} \begin{pmatrix} 1 + \varepsilon_{k,-}^2/|\Delta_k|^2 & 0 \\ 0 & 1 + \varepsilon_{k,+}^2/|\Delta_k|^2 \end{pmatrix}, \tag{219}$$

$$\rightarrow \quad \begin{pmatrix} |u_{k\uparrow\uparrow}|^2 & 0 \\ 0 & |u_{k\downarrow\downarrow}|^2 \end{pmatrix} = \frac{|\Delta_k|^2}{(|\Delta_k|^2 + \varepsilon_{k,+}^2)(|\Delta_k|^2 + \varepsilon_{k,-}^2)} \begin{pmatrix} |\Delta_k|^2 + \varepsilon_{k,-}^2 & 0 \\ 0 & |\Delta_k|^2 + \varepsilon_{k,+}^2 \end{pmatrix}, \tag{220}$$

Assuming $u_{k\uparrow\uparrow}$ and $u_{k\downarrow\downarrow}$ are real and positive,

$$\rightarrow \quad \begin{pmatrix} u_{k\uparrow\uparrow} & 0 \\ 0 & u_{k\downarrow\downarrow} \end{pmatrix} = \frac{|\Delta_k|}{\sqrt{(|\Delta_k|^2 + \varepsilon_{k,+}^2)(|\Delta_k|^2 + \varepsilon_{k,-}^2)}} \begin{pmatrix} \sqrt{|\Delta_k|^2 + \varepsilon_{k,-}^2} & 0 \\ 0 & \sqrt{|\Delta_k|^2 + \varepsilon_{k,+}^2} \end{pmatrix} \tag{221}$$

$$= |\Delta_k| \begin{pmatrix} 1/\sqrt{|\Delta_k|^2 + \varepsilon_{k,+}^2} & 0 \\ 0 & 1/\sqrt{|\Delta_k|^2 + \varepsilon_{k,-}^2} \end{pmatrix} \tag{222}$$

Substituting this into Eq. (148) ($\hat{u}^*_{-k} = \hat{u}_k$),

$$
\begin{aligned}
\hat{v}_k &= \frac{-1}{|\Delta_k|^2}\hat{\Delta}_k\hat{\varepsilon}_{k,+-}\hat{u}^*_{-k} && (223)\\
&= \frac{-1}{|\Delta_k|^2}\begin{pmatrix} -d_x+id_y & 0 \\ 0 & d_x+id_y \end{pmatrix}\begin{pmatrix} \varepsilon_{k,+} & 0 \\ 0 & \varepsilon_{k,-} \end{pmatrix}\begin{pmatrix} u_{k\uparrow\uparrow} & 0 \\ 0 & u_{k\downarrow\downarrow} \end{pmatrix} && (224)\\
&= \frac{-1}{|\Delta_k|^2}\begin{pmatrix} (-d_x+id_y)\varepsilon_{k,+}u_{k\uparrow\uparrow} & 0 \\ 0 & (d_x+id_y)\varepsilon_{k,-}u_{k\downarrow\downarrow} \end{pmatrix} && (225)\\
&= \frac{-1}{|\Delta_k|}\begin{pmatrix} (-d_x+id_y)\varepsilon_{k,+}/\sqrt{|\Delta_k|^2+\varepsilon_{k,+}^2} & 0 \\ 0 & (d_x+id_y)\varepsilon_{k,-}/\sqrt{|\Delta_k|^2+\varepsilon_{k,-}^2} \end{pmatrix}. && (226)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
u_{k\uparrow\uparrow} &= \frac{|\Delta_k|}{\sqrt{|\Delta_k|^2+\varepsilon_{k,+}^2}}, && (227)\\
u_{k\downarrow\downarrow} &= \frac{|\Delta_k|}{\sqrt{|\Delta_k|^2+\varepsilon_{k,-}^2}}, && (228)
\end{aligned}
$$

$$
u_{k\uparrow\downarrow} = u_{k\downarrow\uparrow} = 0, \tag{229}
$$

$$
\begin{aligned}
v_{k\uparrow\uparrow} &= \frac{-(-d_x+id_y)\varepsilon_{k,+}}{|\Delta_k|\sqrt{|\Delta_k|^2+\varepsilon_{k,+}^2}}, && (230)\\
v_{k\downarrow\downarrow} &= \frac{-(d_x+id_y)\varepsilon_{k,-}}{|\Delta_k|\sqrt{|\Delta_k|^2+\varepsilon_{k,-}^2}}, && (231)
\end{aligned}
$$

$$
v_{k\uparrow\downarrow} = v_{k\downarrow\uparrow} = 0. \tag{232}
$$

Here,

$$
\begin{aligned}
\varepsilon_{k,+} &= E_{k,+} - (\xi_k - \mu_{\rm B}H_z) && (233)\\
&= \sqrt{(\xi_k - \mu_{\rm B}H_z)^2 + |\Delta_k|^2} - \xi_k + \mu_{\rm B}H_z, && (234)
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon_{k,-} &= E_{k,-} - (\xi_k + \mu_{\rm B}H_z) && (235)\\
&= \sqrt{(\xi_k + \mu_{\rm B}H_z)^2 + |\Delta_k|^2} - \xi_k - \mu_{\rm B}H_z, && (236)
\end{aligned}
$$

$$
|\Delta_k|^2 = |d_{xk}|^2 + |d_{yk}|^2. \tag{237}
$$

In the case of the unitary triplet state with $\vec{d} \parallel \vec{H}$ (i.e., $\vec{d} = (0, 0, d_z)$),

$$\hat{\Delta}_k^\dagger \hat{\varepsilon}_{k,+-} = \begin{pmatrix} 0 & d_z^* \\ d_z^* & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_{k,+} & 0 \\ 0 & \varepsilon_{k,-} \end{pmatrix} \tag{238}$$

$$= \begin{pmatrix} 0 & d_z^* \varepsilon_{k,-} \\ d_z^* \varepsilon_{k,+} & 0 \end{pmatrix}. \tag{239}$$

$$\hat{\varepsilon}_{k,+-} \hat{\Delta}_k^\dagger \hat{\varepsilon}_{k,+-} = \begin{pmatrix} \varepsilon_{k,+} & 0 \\ 0 & \varepsilon_{k,-} \end{pmatrix} \begin{pmatrix} 0 & d_z^* \varepsilon_{k,-} \\ d_z^* \varepsilon_{k,+} & 0 \end{pmatrix} \tag{240}$$

$$= \begin{pmatrix} 0 & d_z^* \varepsilon_{k,+} \varepsilon_{k,-} \\ d_z^* \varepsilon_{k,+} \varepsilon_{k,-} & 0 \end{pmatrix}. \tag{241}$$

$$\hat{\Delta}_k \hat{\varepsilon}_{k,+-} \hat{\Delta}_k^\dagger \hat{\varepsilon}_{k,+-} = \begin{pmatrix} 0 & d_z \\ d_z & 0 \end{pmatrix} \begin{pmatrix} 0 & d_z^* \varepsilon_{k,+} \varepsilon_{k,-} \\ d_z^* \varepsilon_{k,+} \varepsilon_{k,-} & 0 \end{pmatrix} \tag{242}$$

$$= \begin{pmatrix} |d_z|^2 \varepsilon_{k,+} \varepsilon_{k,-} & 0 \\ 0 & |d_z|^2 \varepsilon_{k,+} \varepsilon_{k,-} \end{pmatrix} \tag{243}$$

$$= |d_z|^2 \varepsilon_{k,+} \varepsilon_{k,-} \hat{\sigma}_0 \tag{244}$$

$$= |\Delta_k|^2 \varepsilon_{k,+} \varepsilon_{k,-} \hat{\sigma}_0. \tag{245}$$

Substituting this into Eq. (162),

$$\hat{u}_k \hat{u}_k^\dagger + \frac{1}{|\Delta_k|^2} \varepsilon_{k,+} \varepsilon_{k,-} \hat{u}_k \hat{u}_k^\dagger = \hat{\sigma}_0, \tag{246}$$

$$\rightarrow \quad \hat{u}_k \hat{u}_k^\dagger = \left[1 + \frac{\varepsilon_{k,+} \varepsilon_{k,-}}{|\Delta_k|^2} \right]^{-1} \hat{\sigma}_0 \tag{247}$$

$$= \left(\frac{1}{|\Delta_k|^2} \right)^{-1} \left[|\Delta_k|^2 + \Big\{ E_{k,+} - (\xi_k - \mu_{\mathrm{B}} H_z) \Big\} \Big\{ E_{k,-} - (\xi_k + \mu_{\mathrm{B}} H_z) \Big\} \right]^{-1} \hat{\sigma}_0 \tag{248}$$

$$= |\Delta_k|^2 \left[|\Delta_k|^2 + \Big\{ E_{0k} - \xi_k \Big\} \Big\{ E_{0k} - \xi_k \Big\} \right]^{-1} \hat{\sigma}_0 \tag{249}$$

$$= |\Delta_k|^2 \Big[|\Delta_k|^2 + \xi_k^2 + E_{0k}^2 - 2E_{0k}\xi_k \Big]^{-1} \hat{\sigma}_0 \tag{250}$$

$$= |\Delta_k|^2 \Big[2E_{0k}^2 - 2E_{0k}\xi_k \Big]^{-1} \hat{\sigma}_0 \tag{251}$$

$$= \frac{|\Delta_k|^2}{2E_{0k}(E_{0k} - \xi_k)} \hat{\sigma}_0 \tag{252}$$

$$= \frac{|d_{zk}|^2}{2E_{0k}(E_{0k} - \xi_k)} \hat{\sigma}_0 \tag{253}$$

$$= \frac{|d_{zk}|^2 (E_{0k} + \xi_k)}{2E_{0k}(E_{0k} - \xi_k)(E_{0k} + \xi_k)} \hat{\sigma}_0 \tag{254}$$

$$= \frac{(E_{0k} + \xi_k)}{2E_{0k}} \hat{\sigma}_0. \tag{255}$$

From this and Eq. (164),

$$(u_k^0 u_k^{z*} + u_k^z u_k^{0*}) = 0, \tag{256}$$

$$|u_k^0|^2 + |u_k^z|^2 = \frac{(E_{0k} + \xi_k)}{2E_{0k}}. \tag{257}$$

Assuming $u_k^z = 0$ and u_k^0 is real, one can determine u_k^0 as

$$u_k^0 = \frac{(E_{0k} + \xi_k)}{\sqrt{2E_{0k}(E_{0k} + \xi_k)}}. \tag{258}$$

Then,

$$\begin{aligned} \hat{u}_k &= u_k^0 \hat{\sigma}_0 \quad (259) \\ &= \frac{(E_{0k} + \xi_k)}{\sqrt{2E_{0k}(E_{0k} + \xi_k)}} \hat{\sigma}_0. \quad (260) \end{aligned}$$

Thus,

$$\begin{aligned} u_{-k}^{0*} &= \frac{(E_{0,-k} + \xi_{-k})}{\sqrt{2E_{0,-k}(E_{0,-k} + \xi_{-k})}} \quad (261) \\ &= \frac{(E_{0k} + \xi_k)}{\sqrt{2E_{0k}(E_{0k} + \xi_k)}}. \quad (262) \end{aligned}$$

And

$$\hat{u}_{-k}^* = \frac{(E_{0k} + \xi_k)}{\sqrt{2E_{0k}(E_{0k} + \xi_k)}} \hat{\sigma}_0. \tag{263}$$

Substituting this into Eq. (148),

$$\begin{aligned} \hat{v}_k &= \frac{-1}{|\Delta_k|^2} \hat{\Delta}_k \hat{\varepsilon}_{k,+-} \hat{u}_{-k}^* \quad (264) \\ &= \frac{-1}{|\Delta_k|^2} \hat{\Delta}_k \hat{\varepsilon}_{k,+-} \frac{(E_{0k} + \xi_k)}{\sqrt{2E_{0k}(E_{0k} + \xi_k)}} \hat{\sigma}_0 \quad (265) \\ &= \frac{-(E_{0k} + \xi_k)}{\sqrt{2E_{0k}(E_{0k} + \xi_k)}} \frac{1}{|d_{zk}|^2} (d_{zk} i\hat{\sigma}_y) \begin{pmatrix} \varepsilon_{k,+} & 0 \\ 0 & \varepsilon_{k,-} \end{pmatrix} \quad (266) \\ &= \frac{-(E_{0k} + \xi_k)}{\sqrt{2E_{0k}(E_{0k} + \xi_k)}} \frac{d_{zk}}{|d_{zk}|^2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_{k,+} & 0 \\ 0 & \varepsilon_{k,-} \end{pmatrix} \quad (267) \\ &= \frac{-(E_{0k} + \xi_k)}{\sqrt{2E_{0k}(E_{0k} + \xi_k)}} \frac{d_{zk}}{|d_{zk}|^2} \begin{pmatrix} 0 & \varepsilon_{k,-} \\ -\varepsilon_{k,+} & 0 \end{pmatrix} \quad (268) \\ &= \frac{-(E_{0k} + \xi_k)}{\sqrt{2E_{0k}(E_{0k} + \xi_k)}} \frac{d_{zk}}{|d_{zk}|^2} \begin{pmatrix} 0 & E_{k,+} - (\xi_k - \mu_{\rm B} H_z) \\ -(E_{k,-} - (\xi_k + \mu_{\rm B} H_z)) & 0 \end{pmatrix} \quad (269) \\ &= \frac{-(E_{0k} + \xi_k)}{\sqrt{2E_{0k}(E_{0k} + \xi_k)}} \frac{d_{zk}}{|d_{zk}|^2} \begin{pmatrix} 0 & (E_{0k} - \xi_k) \\ -(E_{0,k} - \xi_k) & 0 \end{pmatrix} \quad (270) \\ &= \frac{-(E_{0k} + \xi_k)(E_{0k} - \xi_k)}{\sqrt{2E_{0k}(E_{0k} + \xi_k)}} \frac{d_{zk}}{|d_{zk}|^2} i\hat{\sigma}_y \quad (271) \\ &= \frac{-(E_{0k}^2 - \xi_k^2)}{\sqrt{2E_{0k}(E_{0k} + \xi_k)}} \frac{d_{zk}}{|d_{zk}|^2} i\hat{\sigma}_y \quad (272) \\ &= \frac{-|d_{zk}|^2}{\sqrt{2E_{0k}(E_{0k} + \xi_k)}} \frac{d_{zk}}{|d_{zk}|^2} i\hat{\sigma}_y \quad (273) \\ &= \frac{-d_{zk}}{\sqrt{2E_{0k}(E_{0k} + \xi_k)}} i\hat{\sigma}_y. \quad (274) \end{aligned}$$

Hence,

$$u_{k\uparrow\uparrow} = u_{k\downarrow\downarrow} = \frac{(E_{0k} + \xi_k)}{\sqrt{2E_{0k}(E_{0k} + \xi_k)}}, \tag{275}$$

$$u_{k\uparrow\downarrow} = u_{k\downarrow\uparrow} = 0, \tag{276}$$

$$v_{k\uparrow\uparrow} = v_{k\downarrow\downarrow} = 0, \tag{277}$$

$$v_{k\uparrow\downarrow} = -v_{k\downarrow\uparrow} = \frac{-d_{zk}}{\sqrt{2E_{0k}(E_{0k} + \xi_k)}}. \tag{278}$$

Here,

$$E_{0k} = \sqrt{\xi_k^2 + |d_{zk}|^2}. \tag{279}$$

These results for u and v are the same as those in the zero magnetic-field case (Serie 4). Also, the results are the same as those in the singlet state ($d_{zk} \to \Psi_k$).