Unkonventionelle Supraleitung

Serie 9

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9.1 The condensation energy at zero temperature $T=0$. First of all, for the singlet state and the unitary triplet state, the superconducting ground-state energy E_0 at $T=0$ is given by

$$E_0(T=0) = \sum_k \Big(\xi_k - E_k(T=0)\Big) + \frac{1}{2}\sum_{k,s,s'} \frac{|\Delta_{k,ss'}(T=0)|^2}{2E_k(T=0)},$$

where $E_k = \sqrt{\xi_k^2 + |\Delta_k|^2}$ and $|\Delta_k|^2 = \frac{1}{2}\mathrm{Tr}[\hat{\Delta}_k \hat{\Delta}_k^\dagger]$. $s, s' = \{\uparrow, \downarrow\}$. $\Delta_{k,ss'}$ is the (s,s')-th matrix element of $\hat{\Delta}_k$. ξ_k is the energy dispersion in the normal state. The derivation of the above equation, which is rather lengthy, will be shown in *Lösungen*. Instead, here, let us consider the condensation energy by just utilizing the above equation.

The condensation energy F_{cond} at a certain temperature is defined as the difference of the free energies between the superconducting and normal states, $F_{cond} = F_{super} - F_{normal}$. Here, the free energy in the normal state is estimated by setting the superconducting order parameters zero.

At $T=0$, the free energy ($F = E - TS$) is equal to the ground-state energy. One can obtain the ground-state energy in the normal state $E_0^{normal}(T=0)$ from the above equation by setting the order parameters zero.

a) Show that at $T=0$ the condensation energy, $F_{cond} = E_0(T=0) - E_0^{normal}(T=0)$, is given as

$$F_{cond} = \sum_k \Big(|\xi_k| - E_k(T=0)\Big) + \frac{1}{2}\sum_{k,s,s'} \frac{|\Delta_{k,ss'}(T=0)|^2}{2E_k(T=0)}.$$

b) Show that F_{cond} at $T=0$ is given as follows, for the singlet state $\hat{\Delta}_k = \Psi_k i\hat{\sigma}_y$ ($\Psi_k \equiv \Psi_k(T=0)$) and the unitary triplet state $\hat{\Delta}_k = \vec{d}_k \cdot \hat{\vec{\sigma}} i\hat{\sigma}_y$ ($\vec{d}_k \equiv \vec{d}_k(T=0)$), respectively,

$$F_{cond} = -\frac{1}{2}N_0 \int \frac{d\Omega_k}{4\pi}|\Psi_k|^2, \qquad \text{and} \qquad F_{cond} = -\frac{1}{2}N_0 \int \frac{d\Omega_k}{4\pi}|\vec{d}_k|^2.$$

Here, N_0 is the density of states at the Fermi level per spin projection.

Hint: Replace the k summation as

$$\sum_k \to N_0 \int \frac{d\Omega_k}{4\pi}\int_{-\infty}^{\infty} d\xi_k.$$

Introduce the cut-off energy ε_c ($\gg |\Psi_k|, |\vec{d}_k|$):

$$\int_0^{\infty} d\xi_k \to \int_0^{\varepsilon_c} d\xi_k.$$

Assume that Ψ_k and $\vec{d}_k$ do not depend on the energy ξ_k in the k-space, but depends only on the sense of $\vec{k}$ (i.e., on Ω_k). This is the weak-coupling approximation.

There are integration formulas:

$$\int dx\sqrt{x^2+a^2} = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln(x+\sqrt{x^2+a^2}), \qquad \int dx\frac{1}{\sqrt{x^2+a^2}} = \ln(x+\sqrt{x^2+a^2}).$$

9.2 Prove the mathematical formula which appears in Eq. (3.14) of the *German* theory lecture notes:

$$\frac{1}{2\xi}\tanh\Big(\frac{\xi}{2k_\mathrm{B}T}\Big) = 2k_\mathrm{B}T\sum_{m=0}^{\infty}\frac{1}{\omega_m^2+\xi^2},$$

where $\omega_m = \pi k_\mathrm{B}T(2m+1)$.

Hint:

Consider the right hand side.

$$2k_\mathrm{B}T\sum_{m=0}^{\infty}\frac{1}{\omega_m^2+\xi^2} = \frac{1}{\beta}\sum_{m=-\infty}^{\infty}\frac{-1}{(i\omega_m-\xi)(i\omega_m+\xi)} \equiv \frac{1}{\beta}\sum_{m=-\infty}^{\infty}F(i\omega_m).$$

Here, $\beta \equiv 1/k_\mathrm{B}T$. Note that $F(z)$ has the poles at $z=\pm\xi$.

On the other hand, $\exp[\beta(i\omega_n)] = -1$, for the arbitrary integer n. Therefore, for a function $f(z)$,

$$\frac{1}{2\pi i}\int_{C_1} dz\frac{f(z)}{\exp(\beta z)+1} = \frac{-1}{\beta}\sum_{n=-\infty}^{\infty}f(i\omega_n),$$

owing to the residue theorem. Changing the integration path from C_1 to C_2,

$$\frac{1}{2\pi i}\int_{C_1} dz\frac{f(z)}{\exp(\beta z)+1} = \frac{1}{2\pi i}\int_{C_2} dz\frac{f(z)}{\exp(\beta z)+1} = -\sum_{\nu}\frac{R(z_\nu)}{\exp(\beta z_\nu)+1}.$$

Here, z_ν are the poles of the function $f(z)$, and $R(z_\nu)$ is the residue of $f(z)$ at the pole z_ν. Hence,

$$\frac{1}{\beta}\sum_{n=-\infty}^{\infty}f(i\omega_n) = \sum_{\nu}\frac{R(z_\nu)}{\exp(\beta z_\nu)+1}.$$