Unkonventionelle Supraleitung WS 05/06
Lösungen zur Serie 9

9.1 First of all, we will show that the superconducting ground-state energy E_0 at $T = 0$ is given by

$$E_0(T=0) \;=\; \sum_k \Big(\xi_k - E_k(T=0)\Big) + \frac{1}{2}\sum_{k,s,s'} \frac{|\Delta_{k,ss'}(T=0)|^2}{2E_k(T=0)}. \tag{1}$$

Let us start with the following Hamiltonian,

$$H \;=\; \sum_{k,s} \xi_k c^\dagger_{ks} c_{ks} + \frac{1}{2}\sum_{k,k'} \sum_{s_1,s_2,s_3,s_4} V_{k,k';s_1s_2s_3s_4} c^\dagger_{ks_1} c^\dagger_{-ks_2} c_{-k's_3} c_{k's_4}. \tag{2}$$

(See Eq. (2.1) or (55) in the theory lecture notes.) We consider the mean field $b_{k,ss'}$ defined as

$$b_{k,ss'} \equiv \langle c_{-ks} c_{ks'} \rangle, \qquad \text{and} \qquad b^\dagger_{k,ss'} = \langle c^\dagger_{ks'} c^\dagger_{-ks} \rangle \quad (= \langle c_{-ks} c_{ks'} \rangle^* = b^*_{k,ss'}), \tag{3}$$

where $\langle \cdots \rangle$ means the thermal average. We define $\delta b_{k,ss'}$ as

$$\begin{aligned} \delta b_{k,ss'} &\equiv c_{-ks} c_{ks'} - \langle c_{-ks} c_{ks'} \rangle & (4)\\ &= c_{-ks} c_{ks'} - b_{k,ss'}, & (5) \end{aligned}$$

$$\begin{aligned} \delta b^\dagger_{k,ss'} &= (c_{-ks} c_{ks'} - b_{k,ss'})^\dagger & (6)\\ &= c^\dagger_{ks'} c^\dagger_{-ks} - b^\dagger_{k,ss'}. & (7) \end{aligned}$$

Then,

$$c_{-ks} c_{ks'} \;=\; b_{k,ss'} + \delta b_{k,ss'}. \tag{8}$$

$$\begin{aligned} c^\dagger_{ks_1} c^\dagger_{-ks_2} c_{-k's_3} c_{k's_4} &= (c_{-ks_2} c_{ks_1})^\dagger c_{-k's_3} c_{k's_4} & (9)\\ &= (b_{k,s_2s_1} + \delta b_{k,s_2s_1})^\dagger (b_{k',s_3s_4} + \delta b_{k',s_3s_4}) & (10)\\ &= (b^\dagger_{k,s_2s_1} + \delta b^\dagger_{k,s_2s_1})(b_{k',s_3s_4} + \delta b_{k',s_3s_4}) & (11)\\ &= b^\dagger_{k,s_2s_1} b_{k',s_3s_4} + b^\dagger_{k,s_2s_1} \delta b_{k',s_3s_4} + \delta b^\dagger_{k,s_2s_1} b_{k',s_3s_4} + \delta b^\dagger_{k,s_2s_1} \delta b_{k',s_3s_4} & (12)\\ &\approx b^\dagger_{k,s_2s_1} b_{k',s_3s_4} + b^\dagger_{k,s_2s_1} \delta b_{k',s_3s_4} + \delta b^\dagger_{k,s_2s_1} b_{k',s_3s_4} & (13)\\ &= b^\dagger_{k,s_2s_1} b_{k',s_3s_4} + b^\dagger_{k,s_2s_1} (c_{-k's_3} c_{k's_4} - b_{k',s_3s_4}) \\ &\quad + (c_{-ks_2} c_{ks_1} - b_{k,s_2s_1})^\dagger b_{k',s_3s_4} & (14)\\ &= b^\dagger_{k,s_2s_1} b_{k',s_3s_4} + b^\dagger_{k,s_2s_1} (c_{-k's_3} c_{k's_4} - b_{k',s_3s_4}) \\ &\quad + (c^\dagger_{ks_1} c^\dagger_{-ks_2} - b^\dagger_{k,s_2s_1}) b_{k',s_3s_4} & (15)\\ &= b^\dagger_{k,s_2s_1} b_{k',s_3s_4} + b^\dagger_{k,s_2s_1} c_{-k's_3} c_{k's_4} - b^\dagger_{k,s_2s_1} b_{k',s_3s_4} \\ &\quad + c^\dagger_{ks_1} c^\dagger_{-ks_2} b_{k',s_3s_4} - b^\dagger_{k,s_2s_1} b_{k',s_3s_4} & (16)\\ &= b^\dagger_{k,s_2s_1} c_{-k's_3} c_{k's_4} + c^\dagger_{ks_1} c^\dagger_{-ks_2} b_{k',s_3s_4} - b^\dagger_{k,s_2s_1} b_{k',s_3s_4} & (17)\\ &= b^\dagger_{k,s_2s_1} c_{-k's_3} c_{k's_4} + b_{k',s_3s_4} c^\dagger_{ks_1} c^\dagger_{-ks_2} - b^\dagger_{k,s_2s_1} b_{k',s_3s_4}. & (18) \end{aligned}$$

Substituting this into the original Hamiltonian (2), we obtain the mean field Hamiltonian,

$$
\begin{aligned}
H_{MF} &= \sum_{k,s} \xi_k c^\dagger_{ks} c_{ks} \\
&+ \frac{1}{2} \sum_{k,k'} \sum_{s_1,s_2,s_3,s_4} V_{k,k';s_1s_2s_3s_4} (b^\dagger_{k,s_2s_1} c_{-k's_3} c_{k's_4} + b_{k',s_3s_4} c^\dagger_{ks_1} c^\dagger_{-ks_2} - b^\dagger_{k,s_2s_1} b_{k',s_3s_4}) \qquad (19)
\end{aligned}
$$

We define the order parameter as

$$
\Delta_{k,ss'} \equiv - \sum_{k',s_3s_4} V_{k,k';ss's_3s_4} \langle c_{-k's_3} c_{k's_4} \rangle \qquad (20)
$$

$$
= - \sum_{k',s_3s_4} V_{k,k';ss's_3s_4} b_{k',s_3s_4}. \qquad (21)
$$

Then, assuming $V_{k,k';ss's_3s_4}$ is real,

$$
\Delta^*_{k,ss'} = - \sum_{k',s_3s_4} V^*_{k,k';ss's_3s_4} \langle c_{-k's_3} c_{k's_4} \rangle^* \qquad (22)
$$

$$
= - \sum_{k',s_3s_4} V_{k,k';ss's_3s_4} \langle c^\dagger_{k's_4} c^\dagger_{-k's_3} \rangle \qquad (23)
$$

$$
= - \sum_{k',s_3s_4} V_{k,k';ss's_3s_4} b^\dagger_{k',s_3s_4} \qquad (24)
$$

$$
= - \sum_{k',s_3s_4} V_{k',k;s_4s_3s's} b^\dagger_{k',s_3s_4}. \qquad (25)
$$

where we have used the symmetry $V_{k,k';s_1s_2s_3s_4} = V_{k',k;s_4s_3s_2s_1}$ in the last line.[1]

Hence, the mean field Hamiltonian (19) is written as

$$
\begin{aligned}
H_{MF} &= \sum_{k,s} \xi_k c^\dagger_{ks} c_{ks} \\
&+ \frac{1}{2} \sum_{k,k'} \sum_{s_1,s_2,s_3,s_4} V_{k,k';s_1s_2s_3s_4} (b^\dagger_{k,s_2s_1} c_{-k's_3} c_{k's_4} + b_{k',s_3s_4} c^\dagger_{ks_1} c^\dagger_{-ks_2} - b^\dagger_{k,s_2s_1} b_{k',s_3s_4}) \qquad (26) \\
&= \sum_{k,s} \xi_k c^\dagger_{ks} c_{ks} \\
&- \frac{1}{2} \sum_{k,k'} \sum_{s_1,s_2,s_3,s_4} (-V_{k,k';s_1s_2s_3s_4} b^\dagger_{k,s_2s_1}) c_{-k's_3} c_{k's_4} \\
&- \frac{1}{2} \sum_{k,k'} \sum_{s_1,s_2,s_3,s_4} (-V_{k,k';s_1s_2s_3s_4} b_{k',s_3s_4}) c^\dagger_{ks_1} c^\dagger_{-ks_2} \\
&+ \frac{1}{2} \sum_{k,k'} \sum_{s_1,s_2,s_3,s_4} (-V_{k,k';s_1s_2s_3s_4} b^\dagger_{k,s_2s_1}) b_{k',s_3s_4} \qquad (27) \\
&= \sum_{k,s} \xi_k c^\dagger_{ks} c_{ks} \\
&- \frac{1}{2} \sum_{k',s_3,s_4} (- \sum_{k,s_1,s_2} V_{k,k';s_1s_2s_3s_4} b^\dagger_{k,s_2s_1}) c_{-k's_3} c_{k's_4} \\
&- \frac{1}{2} \sum_{k,s_1,s_2} (- \sum_{k',s_3,s_4} V_{k,k';s_1s_2s_3s_4} b_{k',s_3s_4}) c^\dagger_{ks_1} c^\dagger_{-ks_2}
\end{aligned}
$$

[1] M. Sigrist & K. Ueda, Rev. Mod. Phys. **63**, 239 (1991). See *p.* 242 therein.

$$
\begin{aligned}
&+\frac{1}{2}\sum_{k',s_3,s_4}(-\sum_{k,s_1,s_2}V_{k,k';s_1s_2s_3s_4}b^\dagger_{k,s_2s_1})b_{k',s_3s_4} && (28)\\
=\;& \sum_{k,s}\xi_k c^\dagger_{ks}c_{ks}\\
&-\frac{1}{2}\sum_{k',s_3,s_4}\Delta^*_{k',s_4s_3}c_{-k's_3}c_{k's_4}\\
&-\frac{1}{2}\sum_{k,s_1,s_2}\Delta_{k,s_1s_2}c^\dagger_{ks_1}c^\dagger_{-ks_2}\\
&+\frac{1}{2}\sum_{k',s_3,s_4}\Delta^*_{k',s_4s_3}b_{k',s_3s_4} && (29)\\
=\;& \sum_{k,s}\xi_k c^\dagger_{ks}c_{ks}-\frac{1}{2}\sum_{k,s_1,s_2}\left(\Delta^*_{k,s_2s_1}c_{-ks_1}c_{ks_2}+\Delta_{k,s_1s_2}c^\dagger_{ks_1}c^\dagger_{-ks_2}\right)\\
&+\frac{1}{2}\sum_{k,s_1,s_2}\Delta^*_{k,s_2s_1}b_{k,s_1s_2} && (30)\\
=\;& \sum_{k,s}\xi_k c^\dagger_{ks}c_{ks}-\frac{1}{2}\sum_{k,s_1,s_2}\left(\Delta^*_{k,s_2s_1}c_{-ks_1}c_{ks_2}+\Delta_{k,s_1s_2}c^\dagger_{ks_1}c^\dagger_{-ks_2}\right)+C. && (31)
\end{aligned}
$$

Here, we have defined

$$
C\equiv\frac{1}{2}\sum_{k,s_1,s_2}\Delta^*_{k,s_2s_1}b_{k,s_1s_2}. \tag{32}
$$

From Eq. (31), we can write

$$
\begin{aligned}
H_{MF}\;=\;& \sum_{k,s}\xi_k c^\dagger_{ks}c_{ks}-\frac{1}{2}\sum_{k,s_1,s_2}\left(\Delta^*_{k,s_2s_1}c_{-ks_1}c_{ks_2}+\Delta_{k,s_1s_2}c^\dagger_{ks_1}c^\dagger_{-ks_2}\right)+C && (33)\\
=\;& \frac{1}{2}\sum_k\xi_k(c^\dagger_{k\uparrow}c_{k\uparrow}+c^\dagger_{k\downarrow}c_{k\downarrow})+\frac{1}{2}\sum_k\xi_{-k}(c^\dagger_{-k\uparrow}c_{-k\uparrow}+c^\dagger_{-k\downarrow}c_{-k\downarrow})\\
&-\frac{1}{2}\sum_k\Big(\Delta^*_{k,\uparrow\uparrow}c_{-k\uparrow}c_{k\uparrow}+\Delta^*_{k,\downarrow\uparrow}c_{-k\uparrow}c_{k\downarrow}+\Delta^*_{k,\uparrow\downarrow}c_{-k\downarrow}c_{k\uparrow}+\Delta^*_{k,\downarrow\downarrow}c_{-k\downarrow}c_{k\downarrow}\Big)\\
&-\frac{1}{2}\sum_k\Big(\Delta_{k,\uparrow\uparrow}c^\dagger_{k\uparrow}c^\dagger_{-k\uparrow}+\Delta_{k,\uparrow\downarrow}c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow}+\Delta_{k,\downarrow\uparrow}c^\dagger_{k\downarrow}c^\dagger_{-k\uparrow}+\Delta_{k,\downarrow\downarrow}c^\dagger_{k\downarrow}c^\dagger_{-k\downarrow}\Big)+C && (34)\\
=\;& \frac{1}{2}\sum_k\xi_k(c^\dagger_{k\uparrow}c_{k\uparrow}+c^\dagger_{k\downarrow}c_{k\downarrow})+\frac{1}{2}\sum_k\xi_{-k}(2-c_{-k\uparrow}c^\dagger_{-k\uparrow}-c_{-k\downarrow}c^\dagger_{-k\downarrow})\\
&-\frac{1}{2}\sum_k\begin{pmatrix}c_{-k\uparrow}, & c_{-k\downarrow}\end{pmatrix}\begin{pmatrix}\Delta^*_{k,\uparrow\uparrow}c_{k\uparrow}+\Delta^*_{k,\downarrow\uparrow}c_{k\downarrow}\\ \Delta^*_{k,\uparrow\downarrow}c_{k\uparrow}+\Delta^*_{k,\downarrow\downarrow}c_{k\downarrow}\end{pmatrix}\\
&-\frac{1}{2}\sum_k\begin{pmatrix}c^\dagger_{k\uparrow}, & c^\dagger_{k\downarrow}\end{pmatrix}\begin{pmatrix}\Delta_{k,\uparrow\uparrow}c^\dagger_{-k\uparrow}+\Delta_{k,\uparrow\downarrow}c^\dagger_{-k\downarrow}\\ \Delta_{k,\downarrow\uparrow}c^\dagger_{-k\uparrow}+\Delta_{k,\downarrow\downarrow}c^\dagger_{-k\downarrow}\end{pmatrix}+C && (35)\\
=\;& \frac{1}{2}\sum_k\begin{pmatrix}c^\dagger_{k\uparrow}, & c^\dagger_{k\downarrow}\end{pmatrix}\begin{pmatrix}\xi_k & 0\\ 0 & \xi_k\end{pmatrix}\begin{pmatrix}c_{k\uparrow}\\ c_{k\downarrow}\end{pmatrix}+\frac{1}{2}\sum_k\begin{pmatrix}c_{-k\uparrow}, & c_{-k\downarrow}\end{pmatrix}\begin{pmatrix}-\xi_{-k} & 0\\ 0 & -\xi_{-k}\end{pmatrix}\begin{pmatrix}c^\dagger_{-k\uparrow}\\ c^\dagger_{-k\downarrow}\end{pmatrix}\\
&+\frac{1}{2}\sum_k 2\xi_{-k}\\
&-\frac{1}{2}\sum_k\begin{pmatrix}c_{-k\uparrow}, & c_{-k\downarrow}\end{pmatrix}\begin{pmatrix}\Delta^*_{k,\uparrow\uparrow} & \Delta^*_{k,\downarrow\uparrow}\\ \Delta^*_{k,\uparrow\downarrow} & \Delta^*_{k,\downarrow\downarrow}\end{pmatrix}\begin{pmatrix}c_{k\uparrow}\\ c_{k\downarrow}\end{pmatrix}\\
&-\frac{1}{2}\sum_k\begin{pmatrix}c^\dagger_{k\uparrow}, & c^\dagger_{k\downarrow}\end{pmatrix}\begin{pmatrix}\Delta_{k,\uparrow\uparrow} & \Delta_{k,\uparrow\downarrow}\\ \Delta_{k,\downarrow\uparrow} & \Delta_{k,\downarrow\downarrow}\end{pmatrix}\begin{pmatrix}c^\dagger_{-k\uparrow}\\ c^\dagger_{-k\downarrow}\end{pmatrix}+C. && (36)
\end{aligned}
$$

Here, let us define,

$$\vec{A}^\dagger \equiv (\, c_{k\uparrow}^\dagger, \quad c_{k\downarrow}^\dagger \,), \qquad \vec{B}^\dagger \equiv (\, c_{-k\uparrow}, \quad c_{-k\downarrow} \,), \tag{37}$$

$$\vec{A} \equiv \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \end{pmatrix}, \qquad \vec{B} \equiv \begin{pmatrix} c_{-k\uparrow}^\dagger \\ c_{-k\downarrow}^\dagger \end{pmatrix}, \tag{38}$$

$$\hat{\Delta}_k \equiv \begin{pmatrix} \Delta_{k,\uparrow\uparrow} & \Delta_{k,\uparrow\downarrow} \\ \Delta_{k,\downarrow\uparrow} & \Delta_{k,\downarrow\downarrow} \end{pmatrix}, \qquad \hat{\sigma}_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{39}$$

Then, Eq. (36) is

$$\begin{aligned} H_{MF} &= \frac{1}{2}\sum_k (\, c_{k\uparrow}^\dagger, \quad c_{k\downarrow}^\dagger \,) \begin{pmatrix} \xi_k & 0 \\ 0 & \xi_k \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \end{pmatrix} + \frac{1}{2}\sum_k (\, c_{-k\uparrow}, \quad c_{-k\downarrow} \,) \begin{pmatrix} -\xi_{-k} & 0 \\ 0 & -\xi_{-k} \end{pmatrix} \begin{pmatrix} c_{-k\uparrow}^\dagger \\ c_{-k\downarrow}^\dagger \end{pmatrix} \\ &\quad + \frac{1}{2}\sum_k 2\xi_{-k} \\ &\quad - \frac{1}{2}\sum_k (\, c_{-k\uparrow}, \quad c_{-k\downarrow} \,) \begin{pmatrix} \Delta_{k,\uparrow\uparrow}^* & \Delta_{k,\downarrow\uparrow}^* \\ \Delta_{k,\uparrow\downarrow}^* & \Delta_{k,\downarrow\downarrow}^* \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \end{pmatrix} \\ &\quad - \frac{1}{2}\sum_k (\, c_{k\uparrow}^\dagger, \quad c_{k\downarrow}^\dagger \,) \begin{pmatrix} \Delta_{k,\uparrow\uparrow} & \Delta_{k,\uparrow\downarrow} \\ \Delta_{k,\downarrow\uparrow} & \Delta_{k,\downarrow\downarrow} \end{pmatrix} \begin{pmatrix} c_{-k\uparrow}^\dagger \\ c_{-k\downarrow}^\dagger \end{pmatrix} + C \end{aligned} \tag{40}$$

$$\begin{aligned} &= \frac{1}{2}\sum_k \vec{A}^\dagger(\xi_k\hat{\sigma}_0)\vec{A} + \frac{1}{2}\sum_k \vec{B}^\dagger(-\xi_{-k}\hat{\sigma}_0)\vec{B} \\ &\quad + \sum_k \xi_k \\ &\quad - \frac{1}{2}\sum_k \vec{B}^\dagger\hat{\Delta}_k^\dagger\vec{A} \\ &\quad - \frac{1}{2}\sum_k \vec{A}^\dagger\hat{\Delta}_k\vec{B} + C \end{aligned} \tag{41}$$

$$\begin{aligned} &= \frac{1}{2}\sum_k \Big[\vec{A}^\dagger(\xi_k\hat{\sigma}_0)\vec{A} + \vec{B}^\dagger(-\xi_{-k}\hat{\sigma}_0)\vec{B} - \vec{B}^\dagger\hat{\Delta}_k^\dagger\vec{A} - \vec{A}^\dagger\hat{\Delta}_k\vec{B}\Big] \\ &\quad + \sum_k \xi_k + C \end{aligned} \tag{42}$$

$$= \frac{1}{2}\sum_k (\, \vec{A}^\dagger, \quad \vec{B}^\dagger \,) \begin{pmatrix} \xi_k\hat{\sigma}_0 & -\hat{\Delta}_k \\ -\hat{\Delta}_k^\dagger & -\xi_{-k}\hat{\sigma}_0 \end{pmatrix} \begin{pmatrix} \vec{A} \\ \vec{B} \end{pmatrix} + \sum_k \xi_k + C. \tag{43}$$

We define,

$$C_k \equiv \begin{pmatrix} \vec{A} \\ \vec{B} \end{pmatrix} = \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \\ c_{-k\uparrow}^\dagger \\ c_{-k\downarrow}^\dagger \end{pmatrix}, \qquad C_k^\dagger \equiv (\, \vec{A}^\dagger, \quad \vec{B}^\dagger \,) = (\, c_{k\uparrow}^\dagger, \quad c_{k\downarrow}^\dagger, \quad c_{-k\uparrow}, \quad c_{-k\downarrow} \,), \tag{44}$$

$$\check{\varepsilon}_k \equiv \frac{1}{2}\begin{pmatrix} \xi_k\hat{\sigma}_0 & -\hat{\Delta}_k \\ -\hat{\Delta}_k^\dagger & -\xi_{-k}\hat{\sigma}_0 \end{pmatrix}, \tag{45}$$

$$K \equiv \sum_k \xi_k + C, \tag{46}$$

Hence,

$$H_{MF} = \sum_k C_k^\dagger \check{\varepsilon}_k C_k + K. \tag{47}$$

This H_{MF} is essentially equivalent to the Hamiltonian given in Eq. (2.15) and (93) in the theory lecture notes (but notice $\hat{\Delta}_k \to -\hat{\Delta}_k$). Therefore, assuming $\xi_{-k} = \xi_k$, H_{MF} is diagonalized as shown in the theory lecture notes and as in the exercise **4.1**:

$$H_{MF} = \sum_k \left(C_k^\dagger \check{U}_k\right)\left(\check{U}_k^\dagger \check{\varepsilon}_k \check{U}_k\right)\left(\check{U}_k^\dagger C_k\right) + K \tag{48}$$

$$= \sum_k A_k^\dagger \check{E}_k A_k + K, \tag{49}$$

where

$$\check{E}_k = \frac{1}{2}\begin{pmatrix} \hat{E}_k & 0 \\ 0 & -\hat{E}_{-k} \end{pmatrix} \quad \text{and} \quad \hat{E}_k = \begin{pmatrix} E_{k,+} & 0 \\ 0 & E_{k,-} \end{pmatrix}, \tag{50}$$

$$A_k = \begin{pmatrix} a_{k\uparrow} \\ a_{k\downarrow} \\ a_{-k\uparrow}^\dagger \\ a_{-k\downarrow}^\dagger \end{pmatrix}, \tag{51}$$

and $\check{U}_k$ is a unitary matrix.

The diagonalized H_{MF} is then written as

$$H_{MF} = \frac{1}{2}\sum_k \begin{pmatrix} a_{k\uparrow}^\dagger & a_{k\downarrow}^\dagger & a_{-k\uparrow} & a_{-k\downarrow} \end{pmatrix} \begin{pmatrix} E_{k,+} & 0 & 0 & 0 \\ 0 & E_{k,-} & 0 & 0 \\ 0 & 0 & -E_{-k,+} & 0 \\ 0 & 0 & 0 & -E_{-k,-} \end{pmatrix} \begin{pmatrix} a_{k\uparrow} \\ a_{k\downarrow} \\ a_{-k\uparrow}^\dagger \\ a_{-k\downarrow}^\dagger \end{pmatrix} + K \tag{52}$$

$$= \frac{1}{2}\sum_k \Big[E_{k,+}a_{k\uparrow}^\dagger a_{k\uparrow} + E_{k,-}a_{k\downarrow}^\dagger a_{k\downarrow} - E_{-k,+}a_{-k\uparrow}a_{-k\uparrow}^\dagger - E_{-k,-}a_{-k\downarrow}a_{-k\downarrow}^\dagger\Big] + K \tag{53}$$

$$= \frac{1}{2}\sum_k \Big[E_{k,+}a_{k\uparrow}^\dagger a_{k\uparrow} + E_{k,-}a_{k\downarrow}^\dagger a_{k\downarrow} - E_{k,+}a_{k\uparrow}a_{k\uparrow}^\dagger - E_{k,-}a_{k\downarrow}a_{k\downarrow}^\dagger\Big] + K \tag{54}$$

$$= \frac{1}{2}\sum_k \Big[E_{k,+}a_{k\uparrow}^\dagger a_{k\uparrow} + E_{k,-}a_{k\downarrow}^\dagger a_{k\downarrow} - E_{k,+}(1 - a_{k\uparrow}^\dagger a_{k\uparrow}) - E_{k,-}(1 - a_{k\downarrow}^\dagger a_{k\downarrow})\Big] + K \tag{55}$$

$$= \sum_k \Big[E_{k,+}a_{k\uparrow}^\dagger a_{k\uparrow} + E_{k,-}a_{k\downarrow}^\dagger a_{k\downarrow}\Big] - \frac{1}{2}\sum_k (E_{k,+} + E_{k,-}) + K. \tag{56}$$

Since we are now interested in the ground state energy at $T = 0$ where there are no excitations, in Eq. (56) we will consider the following term which corresponds to the ground state energy:

$$E_0 \equiv -\frac{1}{2}\sum_k (E_{k,+} + E_{k,-}) + K \tag{57}$$

$$= -\frac{1}{2}\sum_k (E_{k,+} + E_{k,-}) + \sum_k \xi_k + C \tag{58}$$

$$= -\frac{1}{2}\sum_k (E_{k,+} + E_{k,-}) + \sum_k \xi_k + \frac{1}{2}\sum_{k,s_1,s_2} \Delta_{k,s_2 s_1}^* b_{k,s_1 s_2} \tag{59}$$

$$= -\frac{1}{2}\sum_k (E_{k,+} + E_{k,-}) + \sum_k \xi_k + \frac{1}{2}\sum_{k,s_1,s_2} \Delta_{k,s_2 s_1}^* \langle c_{-ks_1} c_{ks_2} \rangle, \tag{60}$$

where we have referred to Eqs. (3), (32), and (46).

In the case of the singlet state and the unitary triplet state (see Eqs. (2.19)–(2.21) or Eqs. (97)–(99) in the theory lecture notes),

$$E_k \equiv E_{k,+} = E_{k,-} = \sqrt{\xi_k^2 + |\Delta_k|^2} \quad (= E_{-k}), \qquad |\Delta_k|^2 = \frac{1}{2}\mathrm{Tr}[\hat{\Delta}_k \hat{\Delta}_k^\dagger], \tag{61}$$

$$\begin{aligned}
C_k &= \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \\ c_{-k\uparrow}^\dagger \\ c_{-k\downarrow}^\dagger \end{pmatrix} = \check{U}_k A_k = \begin{pmatrix} \hat{u}_k & \hat{v}_k \\ \hat{v}_{-k}^* & \hat{u}_{-k}^* \end{pmatrix} \begin{pmatrix} a_{k\uparrow} \\ a_{k\downarrow} \\ a_{-k\uparrow}^\dagger \\ a_{-k\downarrow}^\dagger \end{pmatrix} && (62) \\
&= \begin{pmatrix} \hat{u}_k \begin{pmatrix} a_{k\uparrow} \\ a_{k\downarrow} \end{pmatrix} + \hat{v}_k \begin{pmatrix} a_{-k\uparrow}^\dagger \\ a_{-k\downarrow}^\dagger \end{pmatrix} \\ \hat{v}_{-k}^* \begin{pmatrix} a_{k\uparrow} \\ a_{k\downarrow} \end{pmatrix} + \hat{u}_{-k}^* \begin{pmatrix} a_{-k\uparrow}^\dagger \\ a_{-k\downarrow}^\dagger \end{pmatrix} \end{pmatrix} && (63) \\
&= \frac{1}{\sqrt{2E_k(E_k+\xi_k)}} \begin{pmatrix} (E_k+\xi_k)\hat{\sigma}_0 \begin{pmatrix} a_{k\uparrow} \\ a_{k\downarrow} \end{pmatrix} + \hat{\Delta}_k \begin{pmatrix} a_{-k\uparrow}^\dagger \\ a_{-k\downarrow}^\dagger \end{pmatrix} \\ \hat{\Delta}_{-k}^* \begin{pmatrix} a_{k\uparrow} \\ a_{k\downarrow} \end{pmatrix} + (E_k+\xi_k)\hat{\sigma}_0 \begin{pmatrix} a_{-k\uparrow}^\dagger \\ a_{-k\downarrow}^\dagger \end{pmatrix} \end{pmatrix} && (64) \\
&= \frac{1}{\sqrt{2E_k(E_k+\xi_k)}} \begin{pmatrix} (E_k+\xi_k)\hat{\sigma}_0 \begin{pmatrix} a_{k\uparrow} \\ a_{k\downarrow} \end{pmatrix} + \hat{\Delta}_k \begin{pmatrix} a_{-k\uparrow}^\dagger \\ a_{-k\downarrow}^\dagger \end{pmatrix} \\ -\hat{\Delta}_k^\dagger \begin{pmatrix} a_{k\uparrow} \\ a_{k\downarrow} \end{pmatrix} + (E_k+\xi_k)\hat{\sigma}_0 \begin{pmatrix} a_{-k\uparrow}^\dagger \\ a_{-k\downarrow}^\dagger \end{pmatrix} \end{pmatrix}, \qquad (\hat{\Delta}_k = -\hat{\Delta}_{-k}^T) && (65) \\
&= \frac{1}{\sqrt{2E_k(E_k+\xi_k)}} \begin{pmatrix} (E_k+\xi_k)a_{k\uparrow} + \Delta_{k,\uparrow\uparrow}a_{-k\uparrow}^\dagger + \Delta_{k,\uparrow\downarrow}a_{-k\downarrow}^\dagger \\ (E_k+\xi_k)a_{k\downarrow} + \Delta_{k,\downarrow\uparrow}a_{-k\uparrow}^\dagger + \Delta_{k,\downarrow\downarrow}a_{-k\downarrow}^\dagger \\ (E_k+\xi_k)a_{-k\uparrow}^\dagger - \Delta_{k,\uparrow\uparrow}^* a_{k\uparrow} - \Delta_{k,\downarrow\uparrow}^* a_{k\downarrow} \\ (E_k+\xi_k)a_{-k\downarrow}^\dagger - \Delta_{k,\uparrow\downarrow}^* a_{k\uparrow} - \Delta_{k,\downarrow\downarrow}^* a_{k\downarrow} \end{pmatrix}. && (66)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle c_{-k\uparrow}c_{k\uparrow}\rangle &= \frac{1}{2E_k(E_k+\xi_k)} \Big\langle \Big[(E_k+\xi_k)a_{-k\uparrow} + \Delta_{-k,\uparrow\uparrow}a_{k\uparrow}^\dagger + \Delta_{-k,\uparrow\downarrow}a_{k\downarrow}^\dagger\Big] \\
&\qquad \times \Big[(E_k+\xi_k)a_{k\uparrow} + \Delta_{k,\uparrow\uparrow}a_{-k\uparrow}^\dagger + \Delta_{k,\uparrow\downarrow}a_{-k\downarrow}^\dagger\Big]\Big\rangle && (67) \\
&= \frac{1}{2E_k(E_k+\xi_k)} \Big[(E_k+\xi_k)\Delta_{k,\uparrow\uparrow}\langle a_{-k\uparrow}a_{-k\uparrow}^\dagger\rangle + (E_k+\xi_k)\Delta_{-k,\uparrow\uparrow}\langle a_{k\uparrow}^\dagger a_{k\uparrow}\rangle\Big] && (68) \\
&= \frac{1}{2E_k}\Big[\Delta_{k,\uparrow\uparrow}\Big\langle(1 - a_{-k\uparrow}^\dagger a_{-k\uparrow})\Big\rangle + \Delta_{-k,\uparrow\uparrow}\langle a_{k\uparrow}^\dagger a_{k\uparrow}\rangle\Big] && (69) \\
&= \frac{1}{2E_k}\Big[\Delta_{k,\uparrow\uparrow}\Big\{1 - f(E_k)\Big\} + \Delta_{-k,\uparrow\uparrow}f(E_k)\Big]. && (70)
\end{aligned}$$

$$\begin{aligned}
\langle c_{-k\uparrow}c_{k\downarrow}\rangle &= \frac{1}{2E_k(E_k+\xi_k)} \Big\langle \Big[(E_k+\xi_k)a_{-k\uparrow} + \Delta_{-k,\uparrow\uparrow}a_{k\uparrow}^\dagger + \Delta_{-k,\uparrow\downarrow}a_{k\downarrow}^\dagger\Big] \\
&\qquad \times \Big[(E_k+\xi_k)a_{k\downarrow} + \Delta_{k,\downarrow\uparrow}a_{-k\uparrow}^\dagger + \Delta_{k,\downarrow\downarrow}a_{-k\downarrow}^\dagger\Big]\Big\rangle && (71) \\
&= \frac{1}{2E_k(E_k+\xi_k)} \Big[(E_k+\xi_k)\Delta_{k,\downarrow\uparrow}\langle a_{-k\uparrow}a_{-k\uparrow}^\dagger\rangle + (E_k+\xi_k)\Delta_{-k,\uparrow\downarrow}\langle a_{k\downarrow}^\dagger a_{k\downarrow}\rangle\Big] && (72) \\
&= \frac{1}{2E_k}\Big[\Delta_{k,\downarrow\uparrow}\Big\langle(1 - a_{-k\uparrow}^\dagger a_{-k\uparrow})\Big\rangle + \Delta_{-k,\uparrow\downarrow}\langle a_{k\downarrow}^\dagger a_{k\downarrow}\rangle\Big] && (73)
\end{aligned}$$

$$= \frac{1}{2E_k}\Big[\Delta_{k,\downarrow\uparrow}\Big\{1 - f(E_k)\Big\} + \Delta_{-k,\uparrow\downarrow} f(E_k)\Big]. \tag{74}$$

$$\begin{aligned}
\langle c_{-k\downarrow} c_{k\uparrow}\rangle &= \frac{1}{2E_k(E_k+\xi_k)}\Big\langle\Big[(E_k+\xi_k)a_{-k\downarrow} + \Delta_{-k,\downarrow\uparrow} a^\dagger_{k\uparrow} + \Delta_{-k,\downarrow\downarrow} a^\dagger_{k\downarrow}\Big] \\
&\qquad \times \Big[(E_k+\xi_k)a_{k\uparrow} + \Delta_{k,\uparrow\uparrow} a^\dagger_{-k\uparrow} + \Delta_{k,\uparrow\downarrow} a^\dagger_{-k\downarrow}\Big]\Big\rangle \quad (75)\\
&= \frac{1}{2E_k(E_k+\xi_k)}\Big[(E_k+\xi_k)\Delta_{k,\uparrow\downarrow}\langle a_{-k\downarrow} a^\dagger_{-k\downarrow}\rangle + (E_k+\xi_k)\Delta_{-k,\downarrow\uparrow}\langle a^\dagger_{k\uparrow} a_{k\uparrow}\rangle\Big] \quad (76)\\
&= \frac{1}{2E_k}\Big[\Delta_{k,\uparrow\downarrow}\Big\langle(1 - a^\dagger_{-k\downarrow} a_{-k\downarrow})\Big\rangle + \Delta_{-k,\downarrow\uparrow}\langle a^\dagger_{k\uparrow} a_{k\uparrow}\rangle\Big] \quad (77)\\
&= \frac{1}{2E_k}\Big[\Delta_{k,\uparrow\downarrow}\Big\{1 - f(E_k)\Big\} + \Delta_{-k,\downarrow\uparrow} f(E_k)\Big]. \quad (78)
\end{aligned}$$

$$\begin{aligned}
\langle c_{-k\downarrow} c_{k\downarrow}\rangle &= \frac{1}{2E_k(E_k+\xi_k)}\Big\langle\Big[(E_k+\xi_k)a_{-k\downarrow} + \Delta_{-k,\downarrow\uparrow} a^\dagger_{k\uparrow} + \Delta_{-k,\downarrow\downarrow} a^\dagger_{k\downarrow}\Big] \\
&\qquad \times \Big[(E_k+\xi_k)a_{k\downarrow} + \Delta_{k,\downarrow\uparrow} a^\dagger_{-k\uparrow} + \Delta_{k,\downarrow\downarrow} a^\dagger_{-k\downarrow}\Big]\Big\rangle \quad (79)\\
&= \frac{1}{2E_k(E_k+\xi_k)}\Big[(E_k+\xi_k)\Delta_{k,\downarrow\downarrow}\langle a_{-k\downarrow} a^\dagger_{-k\downarrow}\rangle + (E_k+\xi_k)\Delta_{-k,\downarrow\downarrow}\langle a^\dagger_{k\downarrow} a_{k\downarrow}\rangle\Big] \quad (80)\\
&= \frac{1}{2E_k}\Big[\Delta_{k,\downarrow\downarrow}\Big\langle(1 - a^\dagger_{-k\downarrow} a_{-k\downarrow})\Big\rangle + \Delta_{-k,\downarrow\downarrow}\langle a^\dagger_{k\downarrow} a_{k\downarrow}\rangle\Big] \quad (81)\\
&= \frac{1}{2E_k}\Big[\Delta_{k,\downarrow\downarrow}\Big\{1 - f(E_k)\Big\} + \Delta_{-k,\downarrow\downarrow} f(E_k)\Big]. \quad (82)
\end{aligned}$$

Hence, the ground state energy E_0 (60) is

$$\begin{aligned}
E_0 &= -\frac{1}{2}\sum_k (E_{k,+} + E_{k,-}) + \sum_k \xi_k + \frac{1}{2}\sum_{k,s_1,s_2} \Delta^*_{k,s_2 s_1}\langle c_{-ks_1} c_{ks_2}\rangle \quad (83)\\
&= -\sum_k E_k + \sum_k \xi_k \\
&\quad + \frac{1}{2}\sum_k \Delta^*_{k,\uparrow\uparrow}\frac{1}{2E_k}\Big[\Delta_{k,\uparrow\uparrow}\Big\{1 - f(E_k)\Big\} + \Delta_{-k,\uparrow\uparrow} f(E_k)\Big] \\
&\quad + \frac{1}{2}\sum_k \Delta^*_{k,\downarrow\uparrow}\frac{1}{2E_k}\Big[\Delta_{k,\downarrow\uparrow}\Big\{1 - f(E_k)\Big\} + \Delta_{-k,\uparrow\downarrow} f(E_k)\Big] \\
&\quad + \frac{1}{2}\sum_k \Delta^*_{k,\uparrow\downarrow}\frac{1}{2E_k}\Big[\Delta_{k,\uparrow\downarrow}\Big\{1 - f(E_k)\Big\} + \Delta_{-k,\downarrow\uparrow} f(E_k)\Big] \\
&\quad + \frac{1}{2}\sum_k \Delta^*_{k,\downarrow\downarrow}\frac{1}{2E_k}\Big[\Delta_{k,\downarrow\downarrow}\Big\{1 - f(E_k)\Big\} + \Delta_{-k,\downarrow\downarrow} f(E_k)\Big] \quad (84)\\
&= -\sum_k E_k + \sum_k \xi_k \\
&\quad + \frac{1}{2}\sum_k \Delta^*_{k,\uparrow\uparrow}\frac{1}{2E_k}\Big[\Delta_{k,\uparrow\uparrow}\Big\{1 - f(E_k)\Big\} - \Delta_{k,\uparrow\uparrow} f(E_k)\Big] \\
&\quad + \frac{1}{2}\sum_k \Delta^*_{k,\downarrow\uparrow}\frac{1}{2E_k}\Big[\Delta_{k,\downarrow\uparrow}\Big\{1 - f(E_k)\Big\} - \Delta_{k,\downarrow\uparrow} f(E_k)\Big] \\
&\quad + \frac{1}{2}\sum_k \Delta^*_{k,\uparrow\downarrow}\frac{1}{2E_k}\Big[\Delta_{k,\uparrow\downarrow}\Big\{1 - f(E_k)\Big\} - \Delta_{k,\uparrow\downarrow} f(E_k)\Big]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_k \Delta^*_{k,\downarrow\downarrow}\frac{1}{2E_k}\Big[\Delta_{k,\downarrow\downarrow}\Big\{1-f(E_k)\Big\}-\Delta_{k,\downarrow\downarrow}f(E_k)\Big], \qquad (\hat{\Delta}_k=-\hat{\Delta}^T_{-k}) \quad (85)\\
=\ &\sum_k(\xi_k-E_k)\\
&+\frac{1}{2}\sum_k \Delta^*_{k,\uparrow\uparrow}\frac{1}{2E_k}\Big[\Delta_{k,\uparrow\uparrow}\Big\{1-2f(E_k)\Big\}\Big]\\
&+\frac{1}{2}\sum_k \Delta^*_{k,\downarrow\uparrow}\frac{1}{2E_k}\Big[\Delta_{k,\downarrow\uparrow}\Big\{1-2f(E_k)\Big\}\Big]\\
&+\frac{1}{2}\sum_k \Delta^*_{k,\uparrow\downarrow}\frac{1}{2E_k}\Big[\Delta_{k,\uparrow\downarrow}\Big\{1-2f(E_k)\Big\}\Big]\\
&+\frac{1}{2}\sum_k \Delta^*_{k,\downarrow\downarrow}\frac{1}{2E_k}\Big[\Delta_{k,\downarrow\downarrow}\Big\{1-2f(E_k)\Big\}\Big] \qquad (86)\\
=\ &\sum_k(\xi_k-E_k)+\frac{1}{2}\sum_{k,s,s'}\frac{|\Delta_{k,ss'}|^2}{2E_k}\tanh\Big(\frac{E_k}{2k_{\mathrm{B}}T}\Big). \qquad (87)
\end{aligned}
$$

At $T=0$,

$$
E_0(T=0) = \sum_k\Big(\xi_k-E_k(T=0)\Big)+\frac{1}{2}\sum_{k,s,s'}\frac{|\Delta_{k,ss'}(T=0)|^2}{2E_k(T=0)}. \qquad (88)
$$

The condensation energy at a certain temperature is defined as the difference of the free energies between the superconducting and normal states. Here, the free energy in the normal state is estimated by setting the superconducting order parameters zero.

At $T = 0$, the free energy is equal to the ground-state energy. One can obtain the ground-state energy in the normal state from Eq. (88) by setting the order parameters zero. Because $E_k = \sqrt{\xi_k^2 + |\Delta_k|^2} \to |\xi_k|$ (for $|\Delta_k| \to 0$),

$$E_0^{normal}(T=0) = \sum_k (\xi_k - |\xi_k|). \tag{89}$$

Therefore, the condensation energy F_{cond} at $T = 0$ is

$$\begin{aligned} F_{cond} &= E_0(T=0) - E_0^{normal}(T=0) && (90)\\ &= \sum_k \Big(\xi_k - E_k(T=0)\Big) + \frac{1}{2}\sum_{k,s,s'} \frac{|\Delta_{k,ss'}(T=0)|^2}{2E_k(T=0)} - \sum_k (\xi_k - |\xi_k|). && (91)\\ &= \sum_k \Big(|\xi_k| - E_k(T=0)\Big) + \frac{1}{2}\sum_{k,s,s'} \frac{|\Delta_{k,ss'}(T=0)|^2}{2E_k(T=0)}. && (92) \end{aligned}$$

In the case of the singlet state $\hat{\Delta}_k = \Psi_k i\hat{\sigma}_y$ ($\Psi_k \equiv \Psi_k(T=0)$),

$$\begin{aligned} |\Delta_k|^2 &= \frac{1}{2}\mathrm{Tr}[\hat{\Delta}_k \hat{\Delta}_k^\dagger] && (93)\\ &= \frac{1}{2}\mathrm{Tr}[(\Psi_k i\hat{\sigma}_y)(-i\hat{\sigma}_y \Psi_k^*)] && (94)\\ &= \frac{|\Psi_k|^2}{2}\mathrm{Tr}[\hat{\sigma}_0] && (95)\\ &= |\Psi_k|^2. && (96) \end{aligned}$$

$$\begin{aligned} E_k &= \sqrt{\xi_k^2 + |\Delta_k|^2} && (97)\\ &= \sqrt{\xi_k^2 + |\Psi_k|^2}. && (98) \end{aligned}$$

$$\begin{aligned} \sum_{s,s'} |\Delta_{k,ss'}(T=0)|^2 &= |\Delta_{k,\uparrow\uparrow}|^2 + |\Delta_{k,\uparrow\downarrow}|^2 + |\Delta_{k,\downarrow\uparrow}|^2 + |\Delta_{k,\downarrow\downarrow}|^2 && (99)\\ &= 0 + |\Psi_k|^2 + |(-\Psi_k)|^2 + 0 && (100)\\ &= 2|\Psi_k|^2. && (101) \end{aligned}$$

Then,

$$\begin{aligned} F_{cond} &= \sum_k \Big(|\xi_k| - E_k(T=0)\Big) + \frac{1}{2}\sum_{k,s,s'} \frac{|\Delta_{k,ss'}(T=0)|^2}{2E_k(T=0)} && (102)\\ &= \sum_k (|\xi_k| - \sqrt{\xi_k^2 + |\Psi_k|^2}) + \frac{1}{2}\sum_k \frac{2|\Psi_k|^2}{2\sqrt{\xi_k^2 + |\Psi_k|^2}} && (103)\\ &= N_0 \int \frac{d\Omega_k}{4\pi} \int_{-\infty}^{\infty} d\xi_k (|\xi_k| - \sqrt{\xi_k^2 + |\Psi_k|^2}) \end{aligned}$$

$$+\frac{1}{2}N_0\int\frac{d\Omega_k}{4\pi}\int_{-\infty}^{\infty}d\xi_k\frac{|\Psi_k|^2}{\sqrt{\xi_k^2+|\Psi_k|^2}} \tag{104}$$

$$= N_0\int\frac{d\Omega_k}{4\pi}2\int_0^{\infty}d\xi_k(\xi_k-\sqrt{\xi_k^2+|\Psi_k|^2}) + \frac{1}{2}N_0\int\frac{d\Omega_k}{4\pi}2\int_0^{\infty}d\xi_k\frac{|\Psi_k|^2}{\sqrt{\xi_k^2+|\Psi_k|^2}}, \tag{105}$$

where N_0 is the density of states at the Fermi level per spin projection.

We now introduce the cut-off energy $\varepsilon_{\rm c}$ ($\gg|\Psi_k|$) in the integration, which shall disappear in the final result. We assume that Ψ_k does not depend on the energy ξ_k in the k-space, but depends only on the sense of $\vec{k}$ (i.e., on Ω_k). That is the weak-coupling approximation. Then,

$$F_{cond} = N_0\int\frac{d\Omega_k}{4\pi}2\int_0^{\infty}d\xi_k(\xi_k-\sqrt{\xi_k^2+|\Psi_k|^2}) + \frac{1}{2}N_0\int\frac{d\Omega_k}{4\pi}2\int_0^{\infty}d\xi_k\frac{|\Psi_k|^2}{\sqrt{\xi_k^2+|\Psi_k|^2}} \tag{106}$$

$$\to N_0\int\frac{d\Omega_k}{4\pi}2\int_0^{\varepsilon_{\rm c}}d\xi_k(\xi_k-\sqrt{\xi_k^2+|\Psi_k|^2}) + \frac{1}{2}N_0\int\frac{d\Omega_k}{4\pi}2\int_0^{\varepsilon_{\rm c}}d\xi_k\frac{|\Psi_k|^2}{\sqrt{\xi_k^2+|\Psi_k|^2}} \tag{107}$$

$$= N_0\int\frac{d\Omega_k}{4\pi}2\left[\frac{\xi_k^2}{2}-\frac{\xi_k}{2}\sqrt{\xi_k^2+|\Psi_k|^2}-\frac{|\Psi_k|^2}{2}\ln\Big(\xi_k+\sqrt{\xi_k^2+|\Psi_k|^2}\Big)\right]_{\xi_k=0}^{\varepsilon_{\rm c}} + \frac{1}{2}N_0\int\frac{d\Omega_k}{4\pi}2|\Psi_k|^2\left[\ln\Big(\xi_k+\sqrt{\xi_k^2+|\Psi_k|^2}\Big)\right]_{\xi_k=0}^{\varepsilon_{\rm c}} \tag{108}$$

$$= N_0\int\frac{d\Omega_k}{4\pi}2\left[\Big\{\frac{\varepsilon_{\rm c}^2}{2}-\frac{\varepsilon_{\rm c}}{2}\sqrt{\varepsilon_{\rm c}^2+|\Psi_k|^2}-\frac{|\Psi_k|^2}{2}\ln\Big(\varepsilon_{\rm c}+\sqrt{\varepsilon_{\rm c}^2+|\Psi_k|^2}\Big)\Big\} - \Big\{-\frac{|\Psi_k|^2}{2}\ln|\Psi_k|\Big\}\right] + \frac{1}{2}N_0\int\frac{d\Omega_k}{4\pi}2|\Psi_k|^2\left[\ln\Big(\varepsilon_{\rm c}+\sqrt{\varepsilon_{\rm c}^2+|\Psi_k|^2}\Big)-\ln|\Psi_k|\right] \tag{109}$$

$$= N_0\int\frac{d\Omega_k}{4\pi}\left[\varepsilon_{\rm c}^2-\varepsilon_{\rm c}\sqrt{\varepsilon_{\rm c}^2+|\Psi_k|^2}-|\Psi_k|^2\ln\Big(\varepsilon_{\rm c}+\sqrt{\varepsilon_{\rm c}^2+|\Psi_k|^2}\Big) + |\Psi_k|^2\ln|\Psi_k|\right] + N_0\int\frac{d\Omega_k}{4\pi}\left[|\Psi_k|^2\ln\Big(\varepsilon_{\rm c}+\sqrt{\varepsilon_{\rm c}^2+|\Psi_k|^2}\Big)-|\Psi_k|^2\ln|\Psi_k|\right] \tag{110}$$

$$= N_0\int\frac{d\Omega_k}{4\pi}\left[\varepsilon_{\rm c}^2-\varepsilon_{\rm c}\sqrt{\varepsilon_{\rm c}^2+|\Psi_k|^2}\ \right] \tag{111}$$

$$= N_0\int\frac{d\Omega_k}{4\pi}\left[\varepsilon_{\rm c}^2-\varepsilon_{\rm c}^2\sqrt{1+\frac{|\Psi_k|^2}{\varepsilon_{\rm c}^2}}\ \right] \tag{112}$$

$$\approx N_0\int\frac{d\Omega_k}{4\pi}\left[\varepsilon_{\rm c}^2-\varepsilon_{\rm c}^2\left(1+\frac{1}{2}\frac{|\Psi_k|^2}{\varepsilon_{\rm c}^2}\right)\ \right] \tag{113}$$

$$= -\frac{1}{2}N_0 \int \frac{d\Omega_k}{4\pi}|\Psi_k|^2. \quad (114)$$

In the case of the unitary triplet state $\hat{\Delta}_k = \vec{d}_k \cdot \hat{\vec{\sigma}} i\hat{\sigma}_y$ ($\vec{d}_k \equiv \vec{d}_k(T=0)$),

$$|\Delta_k|^2 = \frac{1}{2}\mathrm{Tr}[\hat{\Delta}_k\hat{\Delta}_k^\dagger] \quad (115)$$
$$= \frac{1}{2}\mathrm{Tr}[(\vec{d}_k \cdot \hat{\vec{\sigma}} i\hat{\sigma}_y)(-i\hat{\sigma}_y \vec{d}_k^* \cdot \hat{\vec{\sigma}})] \quad (116)$$
$$= \frac{1}{2}\mathrm{Tr}[(\vec{d}_k \cdot \hat{\vec{\sigma}})(\vec{d}_k^* \cdot \hat{\vec{\sigma}})] \quad (117)$$
$$= \frac{1}{2}\mathrm{Tr}\Big[|\vec{d}_k|^2\hat{\sigma}_0 + i(\vec{d}_k \times \vec{d}_k^*) \cdot \hat{\vec{\sigma}}\Big] \quad (118)$$
$$= \frac{1}{2}\mathrm{Tr}\Big[|\vec{d}_k|^2\hat{\sigma}_0\Big] \quad (119)$$
$$= |\vec{d}_k|^2. \quad (120)$$

$$E_k = \sqrt{\xi_k^2 + |\Delta_k|^2} \quad (121)$$
$$= \sqrt{\xi_k^2 + |\vec{d}_k|^2}. \quad (122)$$

$$\sum_{s,s'}|\Delta_{k,ss'}(T=0)|^2 = |\Delta_{k,\uparrow\uparrow}|^2 + |\Delta_{k,\uparrow\downarrow}|^2 + |\Delta_{k,\downarrow\uparrow}|^2 + |\Delta_{k,\downarrow\downarrow}|^2 \quad (123)$$
$$= |-d_x + id_y|^2 + |d_z|^2 + |d_z|^2 + |d_x + id_y|^2 \quad (124)$$
$$= (-d_x + id_y)(-d_x^* - id_y^*) + 2|d_z|^2 + (d_x + id_y)(d_x^* - id_y^*) \quad (125)$$
$$= |d_x|^2 + id_x d_y^* - id_y d_x^* + |d_y|^2$$
$$+ 2|d_z|^2$$
$$+ |d_x|^2 - id_x d_y^* + id_y d_x^* + |d_y|^2 \quad (126)$$
$$= 2(|d_x|^2 + |d_y|^2 + |d_z|^2) \quad (127)$$
$$= 2|\vec{d}_k|^2. \quad (128)$$

Then,

$$F_{cond} = \sum_k\Big(|\xi_k| - E_k(T=0)\Big) + \frac{1}{2}\sum_{k,s,s'}\frac{|\Delta_{k,ss'}(T=0)|^2}{2E_k(T=0)} \quad (129)$$
$$= \sum_k(|\xi_k| - \sqrt{\xi_k^2 + |\vec{d}_k|^2}) + \frac{1}{2}\sum_k\frac{2|\vec{d}_k|^2}{2\sqrt{\xi_k^2 + |\vec{d}_k|^2}}. \quad (130)$$

This is quite the same equation as in the singlet case (Eq. (103)) with just replacing $|\Psi_k| \to |\vec{d}_k|$. Therefore, in the weak-coupling approximation, we finally obtain the final result for the unitary triplet state as in Eq. (114) for the singlet state:

$$F_{cond} = -\frac{1}{2}N_0 \int \frac{d\Omega_k}{4\pi}|\vec{d}_k|^2. \quad (131)$$

9.2 Let us try to prove the mathematical formula which appears in Eq. (3.14) of the *German* theory lecture notes:

$$\frac{1}{2\xi}\tanh\Big(\frac{\xi}{2k_{\mathrm{B}}T}\Big) = 2k_{\mathrm{B}}T\sum_{m=0}^{\infty}\frac{1}{\omega_m^2+\xi^2}, \tag{132}$$

where $\omega_m = \pi k_{\mathrm{B}}T(2m+1)$.

Let us consider the right hand side.

$$\begin{aligned}
2k_{\mathrm{B}}T\sum_{m=0}^{\infty}\frac{1}{\omega_m^2+\xi^2} &= k_{\mathrm{B}}T\sum_{m=-\infty}^{\infty}\frac{1}{\omega_m^2+\xi^2} && (133)\\
&= \frac{1}{\beta}\sum_{m=-\infty}^{\infty}\frac{1}{\xi^2-(i\omega_m)^2} && (134)\\
&= \frac{1}{\beta}\sum_{m=-\infty}^{\infty}\frac{1}{(\xi-i\omega_m)(\xi+i\omega_m)} && (135)\\
&= \frac{1}{\beta}\sum_{m=-\infty}^{\infty}\frac{-1}{(i\omega_m-\xi)(i\omega_m+\xi)} && (136)\\
&\equiv \frac{1}{\beta}\sum_{m=-\infty}^{\infty}F(i\omega_m). && (137)
\end{aligned}$$

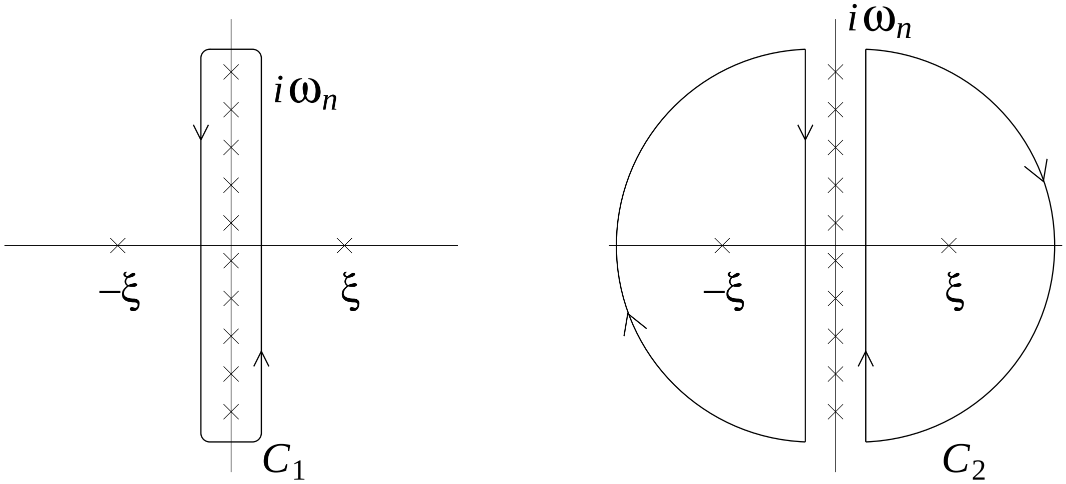
Here, $\beta \equiv 1/k_{\mathrm{B}}T$, and we should note that $F(z)$ has the poles at $z = \pm\xi$.

By the way, for the arbitrary integer n,

$$\begin{aligned}
\exp[\beta(i\omega_n)] &= \exp\Big[\beta\Big(i\frac{\pi}{\beta}(2n+1)\Big)\Big] && (138)\\
&= \exp[i\pi(2n+1)] && (139)\\
&= -1. && (140)
\end{aligned}$$

Therefore, owing to the residue theorem, we can integrate the following for a function $f(z)$,

$$\begin{aligned}
\frac{1}{2\pi i}\int_{C_1}dz\frac{f(z)}{\exp(\beta z)+1} &= \frac{1}{2\pi i}\int_{C_1}dz'\frac{1}{\beta}\cdot\frac{f(z'/\beta)}{\exp(z')+1}, \qquad (z'=\beta z) && (141)\\
&= \frac{1}{2\pi i}\int_{C_1}dz''\frac{1}{\beta z''}\cdot\frac{f\big(\ln(z'')/\beta\big)}{z''+1}, \qquad (z''=\exp(z')=\exp(\beta z)) && (142)\\
&= \sum_{n=-\infty}^{\infty}\lim_{z''\to\exp(\beta i\omega_n)}[z''+1]\frac{1}{\beta z''}\cdot\frac{f\big(\ln(z'')/\beta\big)}{z''+1} && (143)\\
&= \sum_{n=-\infty}^{\infty}\lim_{z\to i\omega_n}[\exp(\beta z)+1]\frac{1}{\beta\exp(\beta z)}\cdot\frac{f(z)}{\exp(\beta z)+1} && (144)\\
&= \sum_{n=-\infty}^{\infty}\lim_{z\to i\omega_n}\frac{1}{\beta\exp(\beta z)}\cdot f(z) && (145)\\
&= \sum_{n=-\infty}^{\infty}\frac{1}{-\beta}\cdot f(i\omega_n) && (146)\\
&= \frac{-1}{\beta}\sum_{n=-\infty}^{\infty}f(i\omega_n). && (147)
\end{aligned}$$



On the other hand, changing the integration path from C_1 to C_2,

$$\frac{1}{2\pi i}\int_{C_1} dz \frac{f(z)}{\exp(\beta z)+1} = \frac{1}{2\pi i}\int_{C_2} dz \frac{f(z)}{\exp(\beta z)+1} \tag{148}$$

$$= -\sum_{\nu} \frac{R(z_\nu)}{\exp(\beta z_\nu)+1}. \tag{149}$$

Here, z_ν are the poles of the function $f(z)$, and $R(z_\nu)$ is the residue of $f(z)$ at the pole z_ν. Hence,

$$\frac{1}{\beta}\sum_{n=-\infty}^{\infty} f(i\omega_n) = \sum_{\nu} \frac{R(z_\nu)}{\exp(\beta z_\nu)+1}. \tag{150}$$

Now, if we set

$$f(z) \equiv F(z) = \frac{-1}{(z-\xi)(z+\xi)}, \tag{151}$$

then the poles of $f(z)$ are at $z = \pm\xi$ and the residues are

$$R(z=\xi) = \lim_{z\to\xi}(z-\xi)F(z) \tag{152}$$

$$= \lim_{z\to\xi} \frac{-1}{(z+\xi)} \tag{153}$$

$$= \frac{-1}{2\xi}, \tag{154}$$

and

$$R(z=-\xi) = \lim_{z\to-\xi}\Big(z-(-\xi)\Big)F(z) \tag{155}$$

$$= \lim_{z\to-\xi} \frac{-1}{(z-\xi)} \tag{156}$$

$$= \frac{1}{2\xi}. \tag{157}$$

Therefore, from Eq. (150),

$$\frac{1}{\beta}\sum_{n=-\infty}^{\infty} f(i\omega_n) = \sum_{\nu} \frac{R(z_\nu)}{\exp(\beta z_\nu)+1}, \tag{158}$$

$$\rightarrow \qquad \frac{1}{\beta}\sum_{m=-\infty}^{\infty} F(i\omega_m) = \frac{R(\xi)}{\exp(\beta\xi)+1} + \frac{R(-\xi)}{\exp(\beta(-\xi))+1} \tag{159}$$

$$= \frac{\frac{-1}{2\xi}}{\exp(\beta\xi)+1} + \frac{\frac{1}{2\xi}}{\exp(-\beta\xi)+1} \tag{160}$$

$$= \frac{-1}{2\xi}\Big(\frac{1}{\exp(\beta\xi)+1} - \frac{1}{\exp(-\beta\xi)+1}\Big) \tag{161}$$

$$= \frac{-1}{2\xi}\Big(\frac{1}{\exp(\beta\xi)+1} - \frac{\exp(\beta\xi)}{1+\exp(\beta\xi)}\Big) \tag{162}$$

$$= \frac{-1}{2\xi}\Big(\frac{1}{\exp(\beta\xi)+1} - \frac{1+\exp(\beta\xi)}{1+\exp(\beta\xi)} + \frac{1}{1+\exp(\beta\xi)}\Big) \tag{163}$$

$$= \frac{-1}{2\xi}\Big(\frac{1}{\exp(\beta\xi)+1} - 1 + \frac{1}{\exp(\beta\xi)+1}\Big) \tag{164}$$

$$= \frac{-1}{2\xi}\Big(\frac{2}{\exp(\beta\xi)+1} - 1\Big) \tag{165}$$

$$= \frac{-1}{2\xi}\Big(\frac{2\exp(-\beta\xi/2)}{\exp(\beta\xi/2)+\exp(-\beta\xi/2)} - 1\Big) \tag{166}$$

$$= \frac{-1}{2\xi}\Big(\frac{2\exp(-\beta\xi/2)}{\exp(\beta\xi/2)+\exp(-\beta\xi/2)} - \frac{\exp(\beta\xi/2)+\exp(-\beta\xi/2)}{\exp(\beta\xi/2)+\exp(-\beta\xi/2)}\Big) \tag{167}$$

$$= \frac{-1}{2\xi}\Big(\frac{\exp(-\beta\xi/2)-\exp(\beta\xi/2)}{\exp(\beta\xi/2)+\exp(-\beta\xi/2)}\Big) \tag{168}$$

$$= \frac{1}{2\xi}\Big(\frac{\exp(\beta\xi/2)-\exp(-\beta\xi/2)}{\exp(\beta\xi/2)+\exp(-\beta\xi/2)}\Big) \tag{169}$$

$$= \frac{1}{2\xi}\tanh\Big(\frac{\beta\xi}{2}\Big) \tag{170}$$

$$= \frac{1}{2\xi}\tanh\Big(\frac{\xi}{2k_{\mathrm{B}}T}\Big). \tag{171}$$

Hence,

$$2k_{\mathrm{B}}T\sum_{m=0}^{\infty}\frac{1}{\omega_m^2+\xi^2} = \frac{1}{\beta}\sum_{m=-\infty}^{\infty} F(i\omega_m) = \frac{1}{2\xi}\tanh\Big(\frac{\xi}{2k_{\mathrm{B}}T}\Big). \tag{172}$$

Q.E.D.